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THE EVALUATION OF THE SPATIAL ACCURACY OF COMPUTER CLASSIFICATION

S.E. PIPER

University of Natal/Department of Surveying and Mapping
Natal, South Africa

SUMMARY

For some time the Remote Sensing Industry has been concerned at the lack of standards for evaluating classifications. It is suggested that a good measure of classification accuracy should be site-specific and should take account of both the errors of omission and commission. Furthermore it is suggested that a "good" measure should be one for which the probability distribution is known and for which one can make statements of statistical significance. The Jaccard co-efficient is proposed and it is shown that it meets the aforementioned criteria. Lastly it is strongly recommended that the precise nature of an analyst's method of measuring accuracy be explicitly stated along with the associated statistical significance.

I. INTRODUCTION

It is part of our remote sensing community's folklore that the aspirant analyst passes through three phases :- euphemism, depression and realism. I would opine that an analyst had reached the third stage when he or she poses the question :- "Just how good is this product?"

There are at least four situations in which a remote sensing analyst would want to assess the quality of a remote sensing product, viz :-

i) A final classification, produced by some semi - or fully-automatic algorithm is to be evaluated in terms of existing surface reference data,

ii) Two classifications, produced by two different techniques (e.g. a 4 - band landsat classification vs a first - two - principal - components classification) are to be compared spatially,

iii) Temporal change is to be assessed from two classifications based on imagery of the same spatial area, but at different dates and

iv) In the geographical data base situation when the analyst is seeking out associations (i.e. The distribution of a herbivor in relation to veld types).

Before proceeding any further it would be wise to define some of the terms introduced above. A classified image, or classification is understood to be a distribution on a two dimension surface of a quantity called the "class" which can take on one of n mutually exclusive and exhaustive values. Furthermore the surface is assumed to be digitized into r rows and c columns, the rectangular elements being called pixels. When comparing two images, or classifications it is assumed that the pixels of each can be brought into a one - to - one relationship with one another. This may lead some readers to think that only rectangular areas may be compared. This is not so. If both classifications are imbedded in a rectangular grid and then multiplied with $(\phi,1)$ mask then areas of any shape may be compared. (The mask being zero everywhere except for those pixels where both classified images are defined, in which case the mask takes on a value of unity). In this way it is possible to compare a classification with surface reference data based on random, or isolated samples. However it is not possible to compare two polygonal areas, unless they have been "rasterized" to lie on the grid.

Before motivating the need for a measure of spatial association it is necessary to consider whether spatial evaluation should be "site-specific", or not, in the sense of Mead and Szajgin (1982). I would argue for site-apecific assesment from three standpoints, viz :-

i) In assessing change or differences the spatial nature of the process should not be ignored,

ii) Site-specific measures are likely to provide a lower limit to non-site-specific measures and as such are more conservative measures (important in an industry noted for its "oversell" !) and

iii) Non-site-specific measures have always offended my sense of mathematical properiety and have left me with the feeling that they are just not Kosher / Halaal.

Having motivated a site-specific measure of association the next question that needs to be proposed is :- "what criteria should a measure of spatial association fulfill?". I would suggest the following five criteria :-

i) The measure should take into account the spatial nature of the agreement and the disagreement,

ii) The measure should be relatively independent of pixel size,

iii) The measure should be independent of the total number of pixels,

iv) The statistically significant critical limits should be available and

v) The measure should be statistically robust.

These criteria will be expanded upon below :-

2. THE MOTIVATION FOR THE JACCARD CO-EFFICIENT.

A. INTRODUCTION

Imagine that there are two classifications to hand. Let the first be an automatic classification with all pixels allocated either to class "A" or to "unclassified". The second shall be Surface Reference Data with all pixels allocated either to class "B" or to "unclassified". If the two images are transparent and are overlayed then the arrangement shown in Fig.1 will result.

There are four distinct areas on the SINS map in Fig.1, viz :-

i) The cross-hatched area of positive association, or "positive matches". The number of pixels in this class is n_{AB},

ii) The horizontally-hatched area of "commissions", i.e. the area where the classification was committed to class "A", but where in fact there was no support from the surface reference data. The number of pixels in this class is n_{Ab}.

iii) The vertically-hatched area of "omissions", i.e. the area where the classification omitted class "A", but where the surface reference data suggested class "B". The number of pixels in this class is n_{aB}.

iv) The blank area, i.e. the area "negative-matches". The number of pixels in this class is n_{ab}.

The relationship between these four classes is :-

$$n_{AB} + n_{Ab} = n_A \qquad (1)$$

$$n_{AB} + n_{aB} = n_B \qquad (2)$$

$$n_{AB} + n_{Ab} + n_{aB} + n_{ab} = N \qquad (3)$$

The relationships are also shown in Table 1. (The name SINS comes from the Penitential Rite:- "That which I have done and that which I have left undone", the sins of commission and omission ! : see English Book of Common Prayer, or the Roman Catholic order of Mass). The SINS table shows the full nature of the spatial relationship between a class (i.e. "A") in one image to a corresponding class (i.e. "B") in a second image.If both images are made up of m classes then it is possible to show the inter-relationships between the two images in a confusion matrix, i.e.Table 2. In a confusion matrix only the positive matches are shown. It is possible to re-compute the confusion matrix and present it as row or column percentages or even with positive matches divided by the total number of pixels,N. Each varient yields a different insight into the nature of the inter-relationship of the two images.

Before giving consideration to the problem of finding a measure of association it is necessary to consider one preliminary question :-

"Are negative matches relelvant ?"

Do they yield any relevant information ? Consider two pairs of classes A and B. If in one classification there are more, or fewer negative matches does that convey any usefull information ? I would suggestnot. Furthermore if the quantities on the right hand sides of equations (1),(2) and (3) are considered fixed (i.e. n_A, n_B and N) then only one of the four remaining quantities (n_{AB}, n_{Ab}, n_{aB} and n_{ab}) may be chosen with complete freedom. Henceforth I shall disregard the negative matches, n_{ab}.

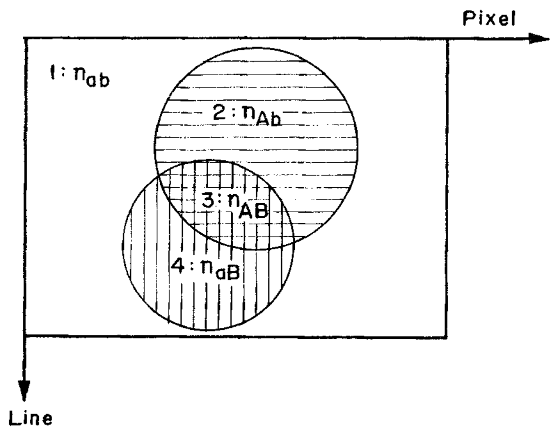
Thus I am seeking some statistic of the form:-

$$J = f(n_{Ab}, n_{aB}, n_{AB})$$

The quantiities n_A, n_B and N are fixed by (i) The total image (for N) (ii) The classification procedure (for n_A) and (iii) The surface reference data collection process (for n_B). While n_{AB} measures the spatial association and n_{Ab}, n_{aB} measure the errors.

Fig. 1. A SINS MAP.

A comparison of Surface Reference Data with a Classified Image.



≡ A. only Surface Reference Data.

|||| B. only Classified Image.

\# Both A & B

The problem faced here is mathematically identical to that found in numerical taxonomy. Sokal and Sneath (1963 : 128 - 141) have reviewed all the 16 possible measures of association between a pair of binary classifications. Applying Occam's razor ("It is vain to do more what can be done with fewer") (Russel 1946 : 463) to this problem leads me to choose the simplest measure, the Jaccaro co-efficient :-

$$J = \frac{n_{AB}}{n_{aB} + n_{Ab} + n_{AB}} \qquad (4)$$

(Occam's razor is the philosophers precaution against Murphy's law). The Jaccard co-efficient has been discussed by Jaccard (1908) and Sneath (1975). It has the following properties :-

i) If there are no positive matches (i.e. $n_{AB} = 0$) then $J = 0$ and

ii) If there is a perfect spatial agreement between A & B, i.e. $n_{AB} = n_A = n_B$ then $J = 1$

iii) If the pixel size is changed by a factor of 2 then the number of pixels will change by (½) and this will not effect J.

Thus we see that J satisfies the first two criteria suggested in section I.Criteria (111)and (1V) (Independence of total number of pixels, N and statistical distribution will be considered together.

B. THE HYPERGEOMETRIC DISTRIBUTION

If the total number of pixels of classes A and B are fixed, and if the total number of pixels N, is fixed then only one of the quantities (n_{AB}, n_{Ab}, n_{aB} and n_{ab}) may be chosen with perfect freedom. Thus I am focusing attention on the spatial arrangement. Using equations (1) and (2) it follows that

$$J = \frac{n_{AB}}{(n_B - n_{AB}) + (n_A - n_{AB}) + n_{AB}} = \frac{n_{AB}}{n_A + n_B - n_{AB}}$$

$$= f\ (n_{AB}) \qquad (5)$$

so J is a function of n_{AB} only as n_A, n_B are fixed. The relationship between J and n_{AB} is shown in Fig.2. From a consideration of equation (5) and an inspection of Fig.2 it may be seen that J and n_{AB} satisfy a one-to-one relationship with each other. Thus if the statistical distribution of n_{AB} were known then the statistical distribution of J would be known. In order to elucidate the nature of the distribution of n_{AB} it is necessary to make an assumption and to formulate a null hypothesis.

Assumption : It is assumed that the surface reference data are spatially specified, *a priori* and may be looked upon as fixed, (i.e. the n_B are fixed)

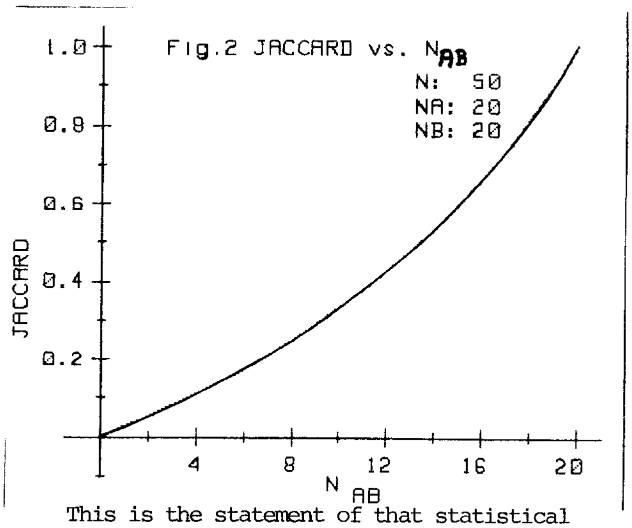
Hypothesis : The classification process allocates the spatial location of the class "A" pixels at random, and does not allocate the same pixel twice.

The clue to finding the statistical distribution is to recognize that the allocation process is the reversal of "selection without replacement". The problem may be formulated in statistical jargon as follows :-

"From a population of N pixels, n_B of whom belong to class B, draw n_A, at random, without replacement. What is the probability that of the sample, size n_A, there will be n_{AB} who are of class B?" Write this as:

$P(X = n_{AB} / n_A, n_B, N)$

with $\mathrm{Max}(\phi, n_A - N + n_B) \leq n_{AB} \leq \mathrm{Min}(n_A, n_B)$ (6)



This is the statement of that statistical sampling process known as the "Hypergeometric distribution". See Kendal and Stuart (1969 Vol 1: 133-135). The major properties of the Hypergeometric distribution are summarized in Appendix 1. The probability mass function is :-

$$P(X = n_{AB} / n_A, n_B, N) = \binom{n_B}{n_{AB}} \cdot \binom{N - n_B}{n_A - n_{AB}} \Big/ \binom{N}{n_{AB}} \qquad (7)$$

This distribution is shown in Fig.3

The minimum value of n_{AB} shown in equation 6 suggests that if both n_A and n_B are relatively large with respect to N then n_{AB} cannot help but be large. This suggests that J is not independent of the total number of pixels, N (i.e. criterion (iii)). However the probability distribution of equation (7) allows for the influence of N. Thus, in turn the probability distribution of J will take N into account.

Testing Hypotheses. The null hypothesis: Hϕ will be as stated above. There are two alternative Hypothesis : HA

(i) The positive association between A and B is so that J will be greater than the J upper critical limit, e.g.:-

$$J > J_{97 \cdot 5\%}$$

(ii) Disassociation between A and B is so great that J will be less than the lower critical limit J_{lower} e.g.

$$J < J_{2 \cdot 5\%}$$

The computation of the critical values is discussed in Appendix 11, and shown graphically in Fig.4

C. BINOMIAL APPROXIMATION

The computation of the critical limits is time-consuming so there exists some incentive to find a computationaly simple approximation. Luckily such an approximation exists, see Appendix I.

The binomial distribution has critical values which can be found in tables for small n_A (See Diem and Lentner 197ϕ) or can be roughly approximated from

$$n_A \cdot \frac{n_B}{N} \pm z_c \cdot (n_A \cdot n_A(N-n_B))^{\frac{1}{2}} / N \quad (8)$$

where z_c is the appropriate unit normal distribution critical value (see Abramowitz and Stegun 1968: 976 - 977)

3. SIMULATION AND RESULTS

In order to test the theory introduced in the previous section a simulation was performed. A small classified image (30 lines by 30 samples, i.e. 900 pixels) with five classes ("Shadow", "Verge", "Grass", "Asphalt" and "Vegetation") was chosen. For each class the mean, median, standard deviation, lower 2½% and upper 97½% confidence limits were computed using both hypergeometric distribution and the binomial approximation.

The simulation, under the null hypothesis, was performed in the following manner :-

i) From the surface reference data the number of pixels in each of the m = 5 classes were counted.

i.e. n_B (i) for classes i = 1,2, ..., m

note that

$$\sum_{i=1}^{m} n_B(i) = N, \text{ the total no. of pixels.}$$

ii) A N x 2 matrix was created. Column 1 contained the class number and column two contained a random number. The first n_B (1) rows were class 1, the second n_B (2) rows class 2 and so on.

iii) Then matrix was sorted on column 2, i.e. sorted by the random number. This effectively randomised the order of the classes.

iv) The first column was then read into an image and became the "classified image".With this classified image it was possible to compute the Jaccard co-efficients for each of the five classes.

By repeating the procedure a number of times estimates for the distribution of the Jaccard coefficient could be obtained.

The results of the simulations based on the hypergeometric and binomial distributions are shown in Table. 3.

The simulation algorithm is described in greater detail in Piper (1983), the evaluation of the Jaccard co-efficient is done as part of the Portable Image Processing Suite (P.I.P.S.). P.I.P.S. is a Univac implementation of V.I.C.A.R. and is described in O'Donoghue et.al. (1983a) and its usage is described in O'Donoghue et.al.(1983b)

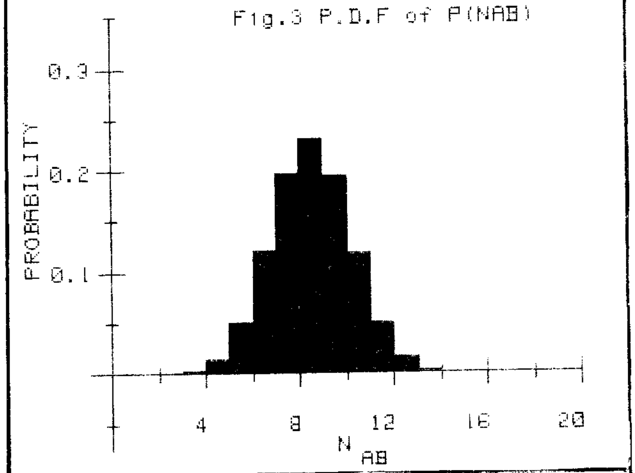


Fig.3 P.D.F of P(NAB)

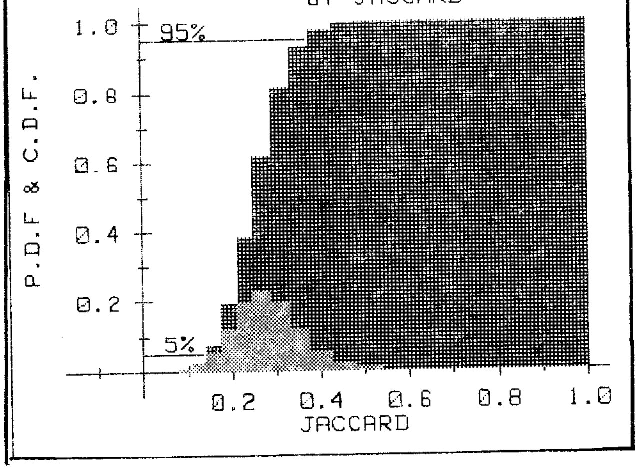


Fig.4 P.D.F & C.D.F. of JACCARD

4. DISCUSSION AND CONCLUSIONS

In the introduction the spatial assessment or comparison of two images was motivated. To many remote sensing analysts the primary concern lies in the evaluation of some automatic, or semiautomatic mapping scheme in terms of a suitable set of Surface Reference Data. It was strongly suggested that all evaluation processes should be site-specific, if for no other reason than that they yield conservative measures.

By considering an analogous problem in numerical taxonomy it was suggested that a parsimonious measure of spatial association, based on the "SINS" map would be the Jaccard co-efficient. Furthermore it was suggested that the desired co-efficient, or measure should satisfy five criteria. The Jaccard co-efficient will now be discussed in the light of these criteria.

(i) Does J take into account the spatial nature of the errors? An examination of equation (4) shows that it measures the effects of both the omissions and commissions. If the measure is to be site-specific then it is necessary that it meets this criterion.

(ii) If the measure were not independent of pixel size then by the simple stratagem of increasing or decreasing the pixel size a co-efficient of any arbitary significance could be generated!

(iii) For the total number of pixels criterion consider the following example:- If the number of pixels of class"A" and of class "B" were both equal to 5ϕ then the overlay could vary from ϕ to 5ϕ if the total number of pixels was 1ϕϕ. However if the total number of pixels was only 75 than the overlap would only range from 25 to 5ϕ. Thus it is necessary to specify that the measure of association be independent of the total number of pixels. Now the Jaccard co-efficient is not independent of the total number of pixels. To satisfy this criterion the statistical distribution of the Jaccard co-efficient was sought. It was shown that J and n_{AB} were in a 1:1 relationship and it was further suggested that n_{AB} followed an hypergeometric distribution. By means of a simulation it has been possible to verify these suggestions. Thus by examining the significance of the Jaccard co-efficient we have a measure which is independent of the total number of pixels, N.

(iv) If the user wants to be sure that a satistically significant association exists between classes "A" and "B" then he needs to be able to compute the critical limits of J. The computation of these units was demonstrated in section 2. The result of this is that if the user finds that $J > J^1$ (where J^1 = upper 97·5% critical value) then he can be 95% certain that class "A" is not randomly distributed with respect to class"B". To translate that "staticulese" into English:- The user can conclude with 95% confidence that class "A" adequately maps class "B".

(v) The last criterion that I suggested was that the measure should be statistically robust. I have not been able to show how sensitive the significance of J is to violations in the assumptions or null hypothesis on which it is based.

In the above it was claimed that the hypergeometric model was adequate. Before proceeding I would like to give consideration to the results in Table 3, and convince the reader that what I claim is true.

The mean value of J is the expected value under the null hypothesis of spatially random classification. It can be seen that the binomial approximation follows the exact hypergeometric distribution while the simulation is a little more variable, but it is not statistically significantly different.

For the critical limits it can be seen that the simulation and hypergeometric results are in good agreement while the binomial approximation tends to be conservatively biased.

In what has gone before I have neglected to give consideration to two problems, viz. :-

i) Physical vs statistical significance.

Table 3 Shows that the 97½% statistical critical value for the class "vegetation" is J = 25·8%. Any analyst accepting so low an association would not attract many users ! Traditionally applied statiticians have used critical values of 5% to 1%. However the remote sensing analyst does not want just to show that an association is not random, he in fact wants to show that it is highly associative. Thus I would suggest that very strict cut-off levels be used, i.e. ϕ·1%, ϕ·ϕϕ1% etc. In this way physical significance will be maintained.

ii) No overall statistic has been suggested. If the user has embarked on a "level 1 classification" with m target classes, how can he measure his overall accuracy. Three possible suggestions are, firstly consider the mean of the J's, secondly the mean of the significance of the J's or thirdly require that each J shall be statistically significant at the ϕ·1% level of significance.

Following out of the above I would offer the following recommendations :-

i) When presenting the results of a mapping/ classification project the analyst should give some measure of his classification accuracy, with respect to relevant surface data,

ii) Measure of accuracy should be site-specific and take both omissions and commissions into account,

iii) When using a measure, of any sort, the analyst should define exactly what quantities were used to calculate that measure, and how the computations were performed,

iv) The statistical significance of a measure of association relative to the null hypothesis of random association should be stated,

v) Minimum levels of statistical significance should be ϕ·5% to ϕ·ϕ1%,

vi) For the overall classification accuracy, of m classes, some overall measure is required,

vii) For the Jaccard co-efficient exact limits may be replaced by the binomial approximation provided the number of pixels n_A, n_B are fairly large,

viii) Some simple set of tables/figures are required for the Jaccard co-efficient to obviate repeated calculations and

ix) The statistical robustness of the Jaccard co-efficient needs to be investigated.

5. ACKNOWLEDGEMENTS

When our former student, Mr.T.P. Boyle returned from a visit to L.A.R.S., Purdue University and other U.S.A., Canadian and English remote sensing centres he was very emphatic in his insistance that we give greater thought to the evaluation of our classifications, and so this project was born. I thank him for his stimulating ideas.

These researchers were first presented to a seminar organized by the the Department of Mathematical Statistics at the University of Durban-Westville, my thanks to them for their kind hearing and especially to Prof. L. Troskie for hinting at the possible use of the Hypergeometric distribution.

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7. NOTATION

The symbols used in this paper are :-

J = The Jaccard co-efficient
m = Total number of different classes
$M = n_B$ = Number of pixels of class "B" in the classified image
$n = n_A$ = Number of pixels of class "A" in the surface reference data
N = Total number of pixels
n_{aB} = Commissions
n_{Ab} = Omissions
n_{ab} = Negative matches
$x = n_{AB}$ = Positive matches
X = A random variable

8. APPENDIX 1

PROPERTIES OF THE HYPERGEOMETRIC DISTIBUTION

The hypergeometric distribution given in equation (7) is more usually written as

$$P(X= x/n,\ N,M) = \binom{M}{x}\cdot\binom{N-M}{n-x} \Big/ \binom{N}{n} \qquad (9)$$

for MAX $(\phi,\ n-N+M) \leq x \leq$ MIN (n,M)

where

$x = n_{AB}$
$n = n_A$ and
$M = n_B$

The binomial co-efficient is

$$\binom{N}{n} = N! \ / \ (\ (N-n)!n!) \qquad (1\phi)$$

The properties of this probability mass function are summarised in Patel et.al. (1976 : 205-207).

The recursive property is

$$P(X=(x+1)/n,N,M) = ((n-x)\ (M-x)\ / \ ((x+1)\ (N-M-n+x+1)))\ \cdot\ P(X=x/n,N,M) \qquad (11)$$

The binomial approximation may be invoked under certain conditions.

put
$$p = M/N \qquad (12)$$
to get
$$P(X=x/n,M,N) = B(X=x/n,p)$$
$$= \binom{n}{x}\ p^x (1-p)^{n-x} \qquad (13)$$

Where B(X=x/n,p) denotes the binomial distribution.

The binomial approximations to the mean and variance would be

$$E\{X\} \doteq nM/N \quad (14)$$

and

$$VAR\{X\} \doteq nM\ (1-M/N)/N = nM(N-M)/N^2 \quad (15)$$

9. APPENDIX 2

COMPUTATION OF THE CUMULATIVE PROBABILITES OF OF THE HYPERGEOMETRIC DISTRIBUTION

The cumulative probability, $Pr(X < A)$ can be defined as

$$Pr(X<A) = \sum_{x=\phi}^{A} Pr(X=x)$$

$$= \sum_{x=\phi}^{A} P(X=x/n,N,M \quad (16)$$

write $F\phi = P(X=\phi/n,N,M) = \binom{M}{\phi}\cdot\binom{N-M}{n-\phi} \Big/ \binom{N}{n}$

$$= \frac{(N-M)!\ (N-n)!n!}{(N-M-n)!\ N!} \quad (17)$$

thus $Pr(X \leq A) = Fo + F_1 + \cdots + Fx$ (18)

use the recursive property of equation (11) to write

$$F_1 = (nM\ /\ (N-M-n+1)\cdot Fo$$

and $F_2 = ((n-1)(M-1)\ /\ 2(N-M-n+2))\ \cdot F_1$

and so on until

$$F_{x+1} = ((n-x)(M-x)/(x+1)(N-M-n+x+1))\cdot Fx \quad (19)$$

In order that computational errors be minimized it is recommended that the recursive co-efficient be written as

$$\frac{n-x}{x+1}\cdot\frac{M-x}{N-M-n+x+1} = \frac{n-x}{x+1}\cdot\frac{M-x}{B+x} \quad (2\phi)$$

where $B = N-M-n+1$ (21)

and compute the quotients first, before multiplying.

Thus the cumulative probability can be evaluated provided $F\phi$ can be computed from equation (17). To prevent machine overflow and/or underflow it is recommended that equation (17) be computed as the product of the three quotients:-

$$F\phi = \frac{(N-M)!}{(N-M-n)!}\cdot\frac{(N-n)!}{N!}\cdot\frac{n!}{1} \quad (22)$$

The factorials may be evaluated directly or read from a look-up table, provided that they are less than 68 for a machine with a dynamic range of

$$\pm 1\phi^{\pm 99}$$

However n,M and N are often larger than this so recourse to an approximation is necessary. Stirlings first order approximation for large x, as (see Abramowitz and Stegan (1968 : 257)

$$Ln(x!) = \tfrac{1}{2}\ Ln(2\pi) + (x+\tfrac{1}{2})Ln(x) - x + \theta/12x \quad (23)$$

with $\phi < \theta < 1$

I will ignore the error term for large x. The natural logarithm of the ratio of two factorials will be

$$Ln\ (x!/y!) = Ln\ (x!) - Ln(y!)$$

$$= xLn(x)+\tfrac{1}{2}Ln(x) - x - yLn(y) - \tfrac{1}{2}Ln(y)+y$$

$$= C\ (say) \quad (24)$$

thus $x!/y! = EXP\ (C)$ (25)

Applying this to equation (17) yields

$$\begin{aligned} F\phi = EXP\ (&(N-M)Ln(N-M) + \tfrac{1}{2}Ln(N-M) - (N-M) \\ &-(N-M-n)Ln(N-M-n) - \tfrac{1}{2}Ln(N-M-n) + (N-M-n) \\ &+(N-n)Ln(N-n) + \tfrac{1}{2}Ln(N-n) - (N-n) \\ &-Nln(N) - \tfrac{1}{2}Ln(N) + N \\ &+ \tfrac{1}{2}Ln(2\pi) + nLn(n) + \tfrac{1}{2}Ln(n)\ -n) \end{aligned}$$

$$\begin{aligned} = EXP\ (&(N-M)Ln(N-M) + \tfrac{1}{2}Ln(N-M) \\ &-(N-M-n)Ln(N-M-n) - \tfrac{1}{2}Ln(N-M-n) \\ &+(N-n)Ln(N-n) + \tfrac{1}{2}Ln(N-n) \\ &-NLn(N) - \tfrac{1}{2}Ln(N) \\ &+nL_n(n) + \tfrac{1}{2}Ln(n) + \tfrac{1}{2}Ln(2\pi)\ -n) \end{aligned} \quad (26)$$

S.E. PIPER has lived nearly all his life in, or near the City of Durban in the Province of Natal, South Africa. He was educated at Kearsney College and read Chemical Engineering at the University of Natal. Thereafter he worked in operations research until moving to the University of the Witwatersrand, Johannesburg, to begin research in mathematical biology from whom he later received his M.Sc. A lecturing post in Applied Mathematics and Environmental Studies at the University of Cape Town then followed before he returned to his hometown. Interludes in a soap factory and as an independent computer consultant preceded his return to his alma mater, first in the Department of Mathematical Statistics and later in Surveying and Mapping. He teaches Geodesy, Remote Sensing, Numerical and Computer Methods, Statistics and Coordinate Systems while his research interests are in Remote Sensing, Geodesy and Ornithology.

TABLE. 1 A 'SINS' TABLE

Classification	Surface Reference Data: Class B	Surface Reference Data: Not B
Class A	n_{AB} positive match	n_{Ab} commission
Not A	n_{aB} omission	n_{ab} negative match

TABLE.2 THE CONFUSION MATRIX

Automatic Classification	Surface Reference Data: Shadow	Verge	Grass	Asphalt	Vege-tation	Total
Class 1	37	3	ϕ	4	6	5ϕ
Class 2	7	82	1	9	8	1ϕ7
Class 3	ϕ	13	91	7	7	118
Class 4	3	2	4	236	34	279
Class 5	3	7	22	23	279	334
Totals	5ϕ	1ϕ7	118	279	334	888

Notes : i) 12 unclassified pixels are not shown
ii) The automatic classification data were simulated so that the totals of corresponding classes balanced.

TABLE.3 A COMPARISON OF THE HYPERGEOMETRIC, BINOMIAL APPROXIMATION AND SIMULATION OF THE JACCARD CO-EFFICIENT

	SHADOW	VERGE	GRASS	ASPHALT	VEGE-TATION
n_A, n_B	5ϕ	1ϕ7	118	279	334
J MEAN : Exact	ϕ·ϕ288	ϕ·ϕ636	ϕ·ϕ7ϕ4	ϕ·1837	ϕ·228ϕ
binomial approximation	ϕ·ϕ286	ϕ·ϕ632	ϕ·ϕ7ϕ2	ϕ·1834	ϕ·2278
simulation	ϕ·ϕ3ϕ8	ϕ·ϕ634	ϕ·ϕ673	ϕ·1879	ϕ·2312
J MEDIAN : Exact	ϕ·ϕ241	ϕ·ϕ594	ϕ·ϕ631	ϕ·1797	ϕ·2257
binomial approximation	ϕ·ϕ286	ϕ·ϕ632	ϕ·ϕ7ϕ2	ϕ·1834	ϕ·2278
simulation	ϕ·ϕ229	ϕ·ϕ613	ϕ·ϕ642	ϕ·1869	ϕ·23ϕ7
J S. D. Exact	ϕ·ϕ168	ϕ ϕ167	ϕ·ϕ167	ϕ·ϕ161	ϕ·ϕ158
simulation	ϕ·ϕ197	ϕ·ϕ171	ϕ·ϕ142	ϕ·ϕ151	ϕ·ϕ159
2½% Critic. val : exact	ϕ·	ϕ·ϕ288	ϕ·ϕ351	ϕ·15ϕ5	ϕ·195ϕ
binominal approximation	ϕ·ϕϕ63	ϕ·ϕ331	ϕ·ϕ4ϕ7	ϕ·1525	ϕ·185ϕ
simulation	ϕ	ϕ·ϕ261	ϕ·ϕ38ϕ	ϕ·16ϕ5	ϕ·1961
97½% Critic. val : exact	ϕ·ϕ526	ϕ·ϕ918	ϕ·ϕ977	ϕ·213ϕ	ϕ·258ϕ
binomial approximation	ϕ·ϕ9ϕ2	ϕ·1ϕ25	ϕ·116ϕ	ϕ·229ϕ	ϕ·269ϕ
simulation	ϕ·ϕ667	ϕ·ϕ947	ϕ·ϕ952	ϕ·23ϕ8	ϕ·2576