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ESTIMATION OF THE LOCATION PARAMETER OF A MULTISPECTRAL DISTRIBUTION BY A MEDIAN OPERATION

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ABSTRACT

Three different kinds of median type estimators for use in applications where the underlying probability distribution is multivariable are proposed and analyzed. Among others such applications arise in the area of remote sensing classification problems, multispectral edges filtering and in computer vision systems. Both aspects, the numerical complexity and the statistical characteristics of the estimators are studied and discussed. From the numerical results obtained by analytical and computer simulation methods there is evidence that a simple extension of the scalar median estimator has an overall performance (in the mean square error sense) that is the same or better than the other two proposed estimators. The computational requirements of this estimator are not excessive making it attractive for applications where large amounts of data are present.

I. INTRODUCTION

Median estimators have been used extensively in a variety of signal processing applications [Frieden, 1976], [Huang, 1979], [Anuta, 1982]. When compared with more conventional estimators such as the sample mean it has been shown [Andrews, 1972] that the median gives a more reliable estimation of the location parameter of impulsive type distributions, i.e. heavy-tailed pdf's (probability distribution functions). When the actual signal contains edges or monotonic changes a moving median estimator has been often proposed as a way to smooth the data and at the same time preserve the edges [Gallagher, 1981]. Most if not all of the applications of the median estimator have been limited to one or two dimensional signals and the measurements collected are basically of 'monochrome' nature, e.g., the kind of data gathered by a monospectral sensor. On the other hand, applications with multispectral signals are already numerous and are growing continuously. Thus we have that most of the data collected by satellites such as the LANDSAT series and the measurements taken by computer vision systems are of multispectral nature.

Location parameter estimators are often used in edge enhancement or filtering. The majority of the procedures developed for the estimation of edges are not suitable for the multispectral nature of the signal in applications such as in Remote Sensing. Edge enhancement techniques, such as the median filter, based on the order statistics of the measured samples can not be easily extended to multispectral data since concepts such as 'greater than' or 'less than', necessary to find the sample order statistics, do not have a meaning for what is basically a vector function.

A common way to get around the multispectral nature of the data has been the use of methods developed for the monospectral case and to apply them separately to each spectral plane. For linear procedures this approach can have some justification in applications where the correlation among the values from different spectral planes is very low. For nonlinear methods such as the median it is not intuitively clear that this approach is reasonable and if it is under which conditions.

The objective of this paper is to propose and to analyze possible extensions of the median estimator to multispectral data. The estimators should keep desirable properties of the scalar version such as edge preserving and robustness against impulsive noise and they should also avoid an excessive increase in the amount of computations required. The analysis is done as much as possible by exact statistical methods. Computer simulation studies are carried out however for cases where an exact analysis is very difficult to obtain.

II. PRELIMINARIES

Let it define a multispectral sample $\{\mathbf{x}_i\}$ as the vectors $\mathbf{x}_1, \ldots \mathbf{x}_N$ being independent random variables with multivariate distributions $\mathbf{f}_1(\mathbf{x})$,

$f_2(\mathbf{x}), \ldots \; f_N(\mathbf{x})$ respectively (note that the random vectors are not necessarily identically distributed). One application where this kind of sample arises is the filtering or enhancement of multispectral edges [Anuta, 1982]. Another application is the classification of homogeneous fields in Remote Sensing data analysis [Kettig, 1975]. In the latter an homogeneous field (a collection of neighboring pixels) is first identified by an 'homogeneity' test on members of the field, then the whole field is classified into one of the possible classes based on a certain distance measure. This distance is computed using a 'mean' vector that represents the field. Difficulties occur when these fields are boundary fields, in which case a few pixels from a different class accepted as members of the field can severely affect the value of the 'mean' vector and thus lead to the whole collection of pixels being wrongly classified. In principle a median type estimator is specially suitable for this kind of data since it is very robust against the presence of 'outliers' in the sample.

A direct straightforward extension of the scalar median estimator is not possible since as indicated earlier there is no equivalent for the sample order statistics in the vector case. In the following section three possible extensions of the median filter to the vector variables are proposed and analized.

III. THE ESTIMATORS

There is an alternate definition of the median filter in the scalar case that can be extended without major difficulty to the vector case. For $\{x_i\}$ a set of scalars, the median can be defined as the value $x_M \in \{x_i\}$ such that the following expression is minimized [Randles, 1979]

$$\sum_{j=1}^{N} |x_j - x_M| \tag{1}$$

It is easy to prove that such value x_M is the conventional median of the set $x_1, x_2, \ldots, x_N$. For the vector variables the following definition is a natural extension of the scalar case. Given the set of vectors $\{\mathbf{x}_i\}$ the median is the vector $\mathbf{x}_{M1} \in \{\mathbf{x}_i\}$ such that

$$\sum_{i=1}^{N} d(\mathbf{x}_i, \mathbf{x}_{M1}) \tag{2}$$

is minimum. $d(\mathbf{x},\mathbf{y})$ is a distance measure between the vectors $\mathbf{x}$ and $\mathbf{y}$. The advantage of this definition is that the estimate obtained is one of the vectors present in the random sample $\{\mathbf{x}_i\}$. However even for simple forms of the distance measure such as $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$ the exact statistical analysis of this estimator is a difficult problem. For the simple case of three independent random vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ with the same multivariate distribution $\mathbf{f}(\mathbf{x})$ the pdf of $\mathbf{x}_{M1}$ is given by the following expression

$$p(\mathbf{x}) = 3\mathbf{f}(\mathbf{x}) \int_{R^n} \int_{R^n(\mathbf{y},\mathbf{z})} \mathbf{f}(\mathbf{y})d\mathbf{y}\mathbf{f}(\mathbf{z})d\mathbf{z} \tag{3}$$

where n is the dimensionality of the vector $\mathbf{x}_i$ (n=number of spectral components) and $R^n(\mathbf{y},\mathbf{z})$ is the region in R^n where $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{y} - \mathbf{z}|$ and $|\mathbf{x} - \mathbf{y}| \leq |\mathbf{z} - \mathbf{y}|$.

To evaluate the perfomance of the proposed estimators the mean square error (mse) criterion is used. Thus for $\mathbf{x}_{M1}$

$$\text{mse}[\mathbf{x}_{M1}] = E[(\mathbf{x}_{M1}-\theta)^T(\mathbf{x}_{M1}-\theta)] \tag{4}$$

where θ is the location parameter to be estimated. It is apparent from (3) that the task of exactly computing the mse of $\mathbf{x}_{M1}$ is a formidable one. For the applications where this estimator is intended to be used the difficulties are increased even more since as mentioned in the previous section the samples $\mathbf{x}_i$'s are not necesarily identically distributed. For this reason in this paper the mse results corresponding to $\mathbf{x}_{M1}$ have been obtained by computer simulation methods.

Since the finding of $\mathbf{x}_{M1}$ involves the computation of all the pairwise distances $d(\mathbf{x}_i,\mathbf{x}_j)$, $i \neq j$, it can be easily shown that the number of operations required by this method is of the order $O(N^2)$.

The second type of median estimator proposed here is the following

$$\mathbf{x}_{M2} = \begin{pmatrix} \text{median } \{x_{i1}\} \\ \text{median } \{x_{i2}\} \\ \cdot \\ \cdot \\ \cdot \\ \text{median } \{x_{in}\} \end{pmatrix} \quad (5)$$

where

$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$ and

$\{x_{ij}\}$ is the set $\{x_{1j}, x_{2j}, \ldots, x_{Nj}\}$

It is obvious from (5) that $\mathbf{x}_{M2}$ is a simple extension of the escalar median estimator. It is also clear that $\mathbf{x}_{M2}$ is not necesarily one of the vectors in $\{\mathbf{x}_i\}$. An advantage of $\mathbf{x}_{M2}$ is that it is possible to carry out an exact statistical analysis of its performance as discussed in the next section.

It is well known that the finding of a scalar median is intimately related to a sorting operation for which algorithms of numerical complexity NlogN have been developed [Knuth, 1975]. From (5) it is apparent that the computational requirements of $\mathbf{x}_{M2}$ is also of the order O(NlogN).

Finally a third kind of vector median estimator is the one based on the **convex hull** of a set of a finite points [Benson, 1966]. This procedure consist in removing, or 'peeling', the convex hull of the set and then removing the convex hull of the remainder continuing until only $(1-2\alpha)N$ points remain [Huber, 1972]. A location parameter estimator can then be obtained by successively removing the convex layers of the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ until the innermost layer has been reached. Define then the median estimator $\mathbf{x}_{M3}$ as the average of the points of this innermost layer. Fig. 1 illustrates this method for a set of points in the plane. Recently an efficient computer algorithm has been proposed which calculates the convex layers of a planar set ($\mathbf{x}_i \in R^2$) in O(NlogN) run time [Chazelle, 1985]. Thus at least for a bispectral case the amount of computation required by $\mathbf{x}_{M3}$ is not excessively large. However as in the case for $\mathbf{x}_{M1}$ the finding of various statistics of $\mathbf{x}_{M3}$ such as the mse is in general very difficult. Computer simulation methods are used here to analyze the statistical behavior of $\mathbf{x}_{M3}$.

It should also be noted that an important feature of the $\mathbf{x}_{M1}$ and $\mathbf{x}_{M3}$ estimators is that both are independent of any translation and rotation of the coordinate axes used to define the vectors $\mathbf{x}_i$, i.e. both yield the same value if the axes are arbitrarily translated and rotated. On the other hand $\mathbf{x}_{M2}$ is by definition intrinsically related to the orientation of the axes and will, in general, give different values if the axes are rotated. This dependence and its effect on the statistical properties of the estimator is dicussed later in the section of numerical results.

IV. STATISTICAL ANALYSIS

As indicated in the previous section the computation of most of the statistics of the proposed estimators is, in general, quite complicated. Even for the simple case of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ assumed to be independent random vectors, identically distributed with pdf $MVN(\mathbf{u}, \Sigma)$ (Multi Variable Normal with mean $\mathbf{u}$ and covariance Σ), the calculation of the mse of estimators such as $\mathbf{x}_{M1}$ and $\mathbf{x}_{M3}$ is indeed difficult.

Since $\mathbf{x}_{M2}$ is a simple generalization of the scalar median it is expected that its statistics are not as hard to compute. The mse of $\mathbf{x}_{M2}$ can be computed in the following manner. For sake of simplicity, assume $\mathbf{x}_i \in R^2$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ distributed as $f_1(x)$ and $\mathbf{x}_{m+1}, \mathbf{x}_{m+2}, \ldots, \mathbf{x}_{m+n}$ distributed as $f_2(x)$ $(m+n=N)$. The computation of statistics such as $E[\mathbf{x}_{M2}^T \mathbf{x}_{M2}]$ can now be easily computed since

$$E[\mathbf{x}_{M2}^T \mathbf{x}_{M2}] = E[(\text{median}\{x_{i1}\})^2 + (\text{median}\{x_{i2}\})^2] \quad (6)$$

Let it be $g_1(x)$ the pdf of median$\{x_{i1}\}$ then $g_1(x)$ is given by [Pomalaza, 1984]

$$g_1(x) = m f_{11}(x) \sum_{j=M-n}^{M} \binom{m-1}{j} \binom{n}{M-j} F_{11}^{j}(x) F_{21}^{M-j}(x)$$

$$[1-F_{11}(x)]^{m-1-j}[1-F_{21}]^{n-M+j} + \qquad (7)$$

$$nf_{21}(x)\sum_{j=M-n+1}^{M}\binom{n-1}{M-j}\binom{m}{j} F_{11}^{j}(x)F_{21}^{M-j}(x)$$

$$[1-F_{11}(x)]^{m-j}[1-F_{21}(x)]^{n-1-M+j}$$

where $M=(N-1)/2$, N is assumed to be odd, and f_{11}, f_{21}, F_{11} and F_{21} are the marginal distributions of $\mathbf{f}_1(\mathbf{x})$, $\mathbf{f}_2(\mathbf{x})$, $\mathbf{F}_1(\mathbf{x})$ and $\mathbf{F}_2(\mathbf{x})$ respectively. The pdf of median$\{x_{i2}\}$ can be also computed in the same manner. Once the pdf's are known the computation of (6) is usually a straightforward numerical integration.

As mentioned previously the value of $\mathbf{x}_{M2}$ is dependent on the orientation of the coordinate axes. On the other hand, expected values of functions of $\mathbf{x}_{M2}$ can in some cases be independent of the axes orientations. Thus for example if $\mathbf{f}_1(x)$ and $\mathbf{f}_2(x)$ are MVN with the same mean vector and covariance matrix then it can be proved that (6) is the same under any rotation of the coordinate axes.

The results discussed in the next section are obtained by numerical integration techniques ($\mathbf{x}_{M2}$) and by computer simulation methods ($\mathbf{x}_{M1}$ and $\mathbf{x}_{M3}$). The perfomance of a more conventional location parameter estimator is also computed. This estimator is the sample mean $\mathbf{x}_A$ and is defined as

$$\mathbf{x}_A=(1/N)\sum_{i=1}^{N}\mathbf{x}_i \qquad (8)$$

Since $\mathbf{x}_A$ is much easier to compute than any of the estimators proposed here, the only cases of interest are the ones for which the mse of $\mathbf{x}_{M1}$, (or $\mathbf{x}_{M2}$ or $\mathbf{x}_{M3}$) is smaller than the mse of $\mathbf{x}_A$.

V. NUMERICAL RESULTS

To illustrate the performance of the proposed estimators the following example is considered. Let it assume a sample of N=9 independent random vectors $\mathbf{x}_i \in R^2$ (bispectral data). $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$ have a multivariate distribution $\mathbf{f}_1(\mathbf{x})=MVN(0, \Sigma_1)$, and $\mathbf{x}_{m+1},\ldots,\mathbf{x}_{m+n}$ have a distribution $\mathbf{f}_2(\mathbf{x})=MVN(\mathbf{h}, \Sigma_2)$ (m+n=N and m > n). This sample could for example arise in an edge filtering application when a moving window of 3x3 is moving across a bispectral edge of height $\mathbf{h}$. Here it is wanted to estimate the signal value that corresponds to the center pixel of the window. Assuming that the intersection of the edge with the window is a straight line then the parameter to be estimated in this example is the distribution mean of $\mathbf{x}_1,\ldots, \mathbf{x}_m$, i.e. zero.

To consider different cases for $\mathbf{h}$, Σ_1 and Σ_2, it is initially assumed that both covariance matrices are equal to Λ and $\mathbf{R}\Lambda$ respectively, where Λ is a diagonal matrix and $\mathbf{R}$ a linear transformation (rotation by an angle ϕ_2). In this manner by varying ϕ_2, different orientations of the eigenvectors of Σ_1 with respect to the eigenvectors of Σ_2 can be obtained as illustrated in Fig. 2 (a). The effect of the axis orientation on the value of $\mathbf{x}_{M2}$ is studied by applying a linear transformation (rotation by an angle ϕ_1) to $\mathbf{x}_1,\ldots,\mathbf{x}_N$. This results in a change of the distributions as shown in Fig. 2 (b). Values of ϕ_1 considered are 0, $\pi/8$ and $\pi/4$. Due to the symmetry of the problem results for other angles such as $3\pi/4$, π,etc. can be easily inferred from the ones already computed.

Table 1 (a)-(c) summarizes the mse values of the various estimators discussed in this paper for the case of m=6, n=3 and Λ=diag[9,1]. For the scalar case, i.e. monospectral edge, it has been shown [Justusson, 1979], [Pomalaza, 1980] that in general, the median gives a smaller mse than the sample mean, for edge heights greater than twice the standard deviation σ. There it is assumed that m samples have $N(0,\sigma^2)$ distribution and the remaining n samples have $N(h,\sigma^2)$ distribution. From the results on Table 1 (a)-(c) it can be said that for the bispectral example discussed here, the various kinds of median operators proposed have a better perfomance than the sample mean whenever $|\mathbf{h}|$ is roughly equal to twice $\sqrt{\lambda_1}+\sqrt{\lambda_2}$ or greater. λ_1 and λ_2 are the eigenvalues of which in this case are 9 and 1 respectively. Since the cases shown in Table 1 cover a wide variety of cross-correlation values among the samples components it is fair to expect that the same type of perfomance is obtained for different values of λ_1 and λ_2.

When comparing among the various median estimators it becomes apparent that x_{M2} has an overall better perfomance than x_{M1} and x_{M3} since all of them have roughly the same perfomance for values of $|h|$ comparable to twice the sum $\sqrt{\lambda_1}+\sqrt{\lambda_2}$, and x_{M2} is definitely better (in the mse sense) than x_{M1} or x_{M3} for larger values of $|h|$.

Tables 2 (a)-(c) and 3 (a)-(c) shows the results for m=5,n=4 and m=7,n=2 respectively. Table 2 corresponds to the case of worst 'contamination' for which x_{M2} is not clearly better than x_{M1} and x_{M3}. This is specially true for the results shown in Table 2 (c) where x_{M2} seems to be very sensitive to the coordinate axis orientation and x_{M1} appears to be a better choice. Table 3 corresponds to the least amount of contamination than the previous two cases and x_{M2} is definitely more atractive than x_{M1} or x_{M3}. It is interesting to note that x_{M3} has been often proposed as an extension to the scalar median for the multidimensional case [Shamos, 1978]. However its statistical behavior (mse), at least for the case discussed in this section, doesn't seem to be very good.

Since the computational requirements of x_{M2} are not worse than x_{M1} and x_{M3} it is fair to conclude that for applications where a median type estimator is wanted, e.g. multispectral edge enhancement, the use of x_{M1} is a reasonable procedure. There are of course applications, e.g. filtering of multispectral lines, where as in the scalar case no median type operation will give good results and different kind of algorithms have to be proposed [Pomalaza, 1984].

VI. CONCLUSIONS

In this paper three possible median type estimators for multispectral data are proposed and analyzed. Areas where these estimators are intended to be applied are many, such as multispectral edge filtering or enhancement and classification of homegeneous fields in Remote Sensing data analysis. Numerical results obtained by exact analytical methods and computer simulation give evidence that the estimator that is the most simple to define and to compute has also and overall better perfomance for a variety of data configurations.

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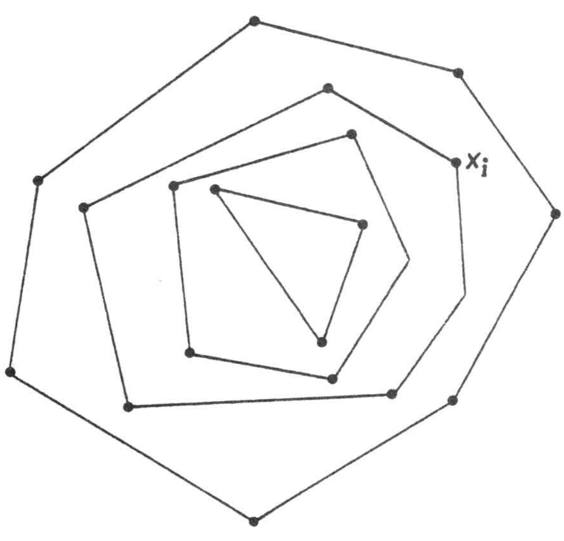

Figure 1. The convex layers of a set of planar points. $\mathbf{x}_{M3}$ is defined as the average of the innermost layer points.

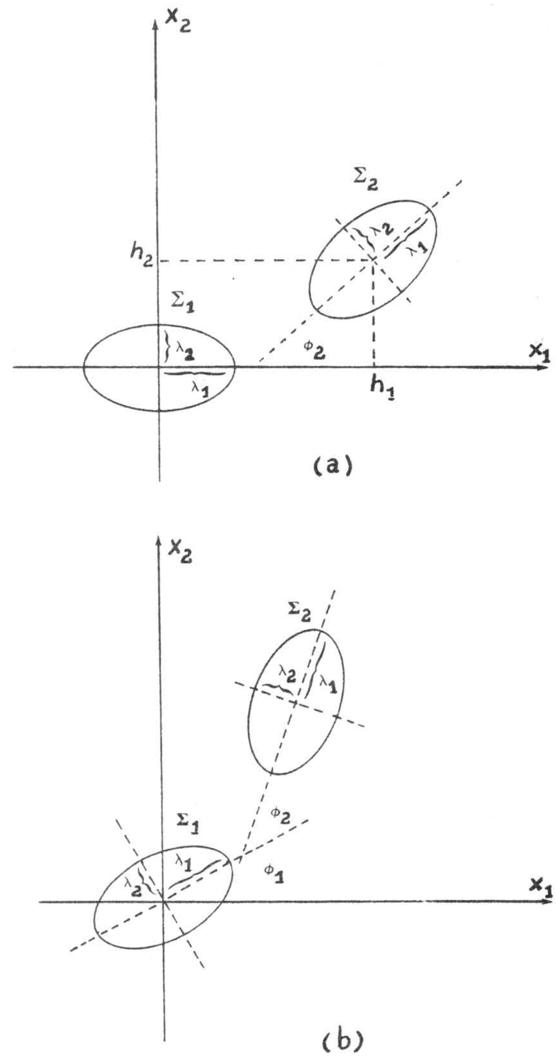

Figure 2. Ellipses of equi-probability corresponding to the bivariate normal distributions $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$.

Table 1

Mean Square Error
$m=6,\ n=3$

(a) $\phi_2=0$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.111	5.111	10.111
x_{M1}	2.806	5.833	10.086
$x_{M2}{}^{(1)}$	2.697	4.790	6.056
$x_{M2}{}^{(2)}$	2.703	4.885	6.361
$x_{M2}{}^{(3)}$	2.706	4.915	6.420
x_{M3}	2.793	5.552	8.810

(b) $\phi_2=\pi/4$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.111	5.111	10.111
x_{M1}	2.969	5.759	9.790
$x_{M2}{}^{(1)}$	2.915	5.416	6.370
$x_{M2}{}^{(2)}$	3.088	5.624	6.352
$x_{M2}{}^{(3)}$	2.946	5.310	6.379
x_{M3}	2.878	5.338	7.642

(c) $\phi_2=\pi/2$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.111	5.111	10.111
x_{M1}	3.001	5.713	9.785
$x_{M2}{}^{(1)}$	3.618	6.238	6.498
$x_{M2}{}^{(2)}$	3.088	5.624	6.352
$x_{M2}{}^{(3)}$	2.706	4.915	6.420
x_{M3}	2.860	5.209	7.285

(1) $\phi_1=0$, (2) $\phi_1=\pi/8$, (3) $\phi_1=\pi/4$

Table 2

Mean Square Error
$m=5,\ n=4$

(a) $\phi_2=0$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.888	8.222	17.111
x_{M1}	3.590	9.070	16.935
$x_{M2}{}^{(1)}$	3.535	8.460	13.403
$x_{M2}{}^{(2)}$	3.538	8.538	13.930
$x_{M2}{}^{(3)}$	3.539	8.572	14.108
x_{M3}	3.636	9.241	17.730

(b) $\phi_2=\pi/4$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.888	8.222	17.111
x_{M1}	3.826	9.194	14.559
$x_{M2}{}^{(1)}$	3.888	9.980	14.787
$x_{M2}{}^{(2)}$	4.145	10.722	15.012
$x_{M2}{}^{(3)}$	3.904	9.798	14.577
x_{M3}	3.800	9.049	15.887

(c) $\phi_2=\pi/2$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	2.888	8.222	17.111
x_{M1}	3.988	8.931	13.824
$x_{M2}{}^{(1)}$	5.006	12.944	16.018
$x_{M2}{}^{(2)}$	4.145	10.722	15.012
$x_{M2}{}^{(3)}$	3.539	8.572	14.108
x_{M3}	3.812	8.789	14.809

(1) $\phi_1=0$, (2) $\phi_1=\pi/8$, (3) $\phi_1=\pi/4$

Table 3

Mean Square Error
m=7, n=2

(a) $\phi_2=0$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	1.555	2.888	5.111
x_{M1}	2.237	3.640	5.725
$x_{M2}^{(1)}$	2.128	2.881	3.206
$x_{M2}^{(2)}$	2.133	2.935	3.335
$x_{M2}^{(3)}$	2.135	2.949	3.351
x_{M3}	2.278	3.416	4.387

(b) $\phi_2=\pi/4$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	1.555	2.888	5.111
x_{M1}	2.312	3.624	5.685
$x_{M2}^{(1)}$	2.228	3.106	3.303
$x_{M2}^{(2)}$	2.320	3.153	3.296
$x_{M2}^{(3)}$	2.258	3.056	3.321
x_{M3}	2.224	3.250	3.693

(c) $\phi_2=\pi/2$

	h=(3,0)	h=(6,0)	h=(9,0)
x_A	1.555	2.888	5.111
x_{M1}	2.333	3.559	5.585
$x_{M2}^{(1)}$	2.566	3.307	3.097
$x_{M2}^{(2)}$	2.320	3.153	3.296
$x_{M2}^{(3)}$	2.135	2.949	3.351
x_{M3}	2.236	3.109	3.498

(1) $\phi_1=0$, (2) $\phi_1=\pi/8$, (3) $\phi_1=\pi/4$

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