Reprinted from

Eleventh International Symposium

Machine Processing of

Remotely Sensed Data

with special emphasis on

Quantifying Global Process: Models, Sensor Systems, and Analytical Methods

June 25 - 27, 1985

Proceedings

Purdue University
The Laboratory for Applications of Remote Sensing
West Lafayette, Indiana 47907 USA

RANGE OF VALIDITY OF TAYLOR SERIES APPROACH TO VARIANCE OF REGISTRATION ERROR

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ABSTRACT

The performance of a registration system is often measured using the variance of registration error. The variance of registration error is usually not found exactly but approximated through small signal analysis of the registration system model. The accuracy of the variance of registration error obtained in this way is often in question.

This paper develops a procedure for finding upper and lower bounds on the variance of registration error. The procedure is applied to a one dimensional, discrete registration system. Results are presented for signals with a variety of different spectra.

I. INTRODUCTION

There are many situations were one must first align or register two signals before the two signals can be processed. For example noisy signals from the sensors in a beam detection array are often first aligned or registered and then summed to form the desired composite signal.

The procedures used for affecting signal registration differ in detail but in general involve computation of some kind of matching function which peaks in the vicinity of the correct registration position. The signals involved are usually corrupted with noise. The noise affects the position for which the matching function peaks, thereby causing registration errors. In practical systems the registration error is not known exactly and therefore must be treated as a random variable.

A measure of registration system performance that is commonly used is the variance of registration error. For most registration systems the variance of registration error can not be obtained exactly so it must be estimated or obtained through approximation. One method of obtaining an approximate variance of registration error involves approximating the portion of the matching function that is independent of the noise, in the vicinity of its peak, with the first few terms of its Taylor series expansion. Analysis of the approximated matching function yields a variance, which approximates the variance of registration error for the original matching function. This method was used by Helstrom[2] in a continuous radar range estimation problem. There are a number of examples[1][5] where this method has been applied to one dimensional problems. McGillem and Svedlow[4] extended this method to obtain an approximation for the variance of registration error for a two dimensional continuous signal registration system. Steding and Smith[6] applied this approach to obtain a variance of registration error for a two dimensional registration model which differed slightly from that of McGillem and Svedlow.

A major difficulty in using the variance found by the above method is uncertainty about whether or not the approximation is accurate. It is sure to be accurate if the registration error is "small" but then the question of what constitutes "small" arises. Currently, the only quantitative guide is due to Ianniello[3] who studied the threshold effect that occurs in the registration problem when signal to noise ratio becomes small. This threshold effect is a consequence of the possible occurrence of large registration errors caused by "false peaks" in the matching function which occur at low signal to noise ratios. The term "false peak" is used to refer to a peak occurring at a matching position substantially removed from the main lobe of the matching function. Ianniello has provided a quantitative result for predicting the onset of threshold for the one dimensional case when a cross-correlation function is used as a matching function. Since the Taylor series approximation tends to eliminate any false peaks in the matching function, the variance of registration error based on this approximation will be very optimistic for signal to noise ratios (SNRs) below threshold. In fact one might anticipate poor accuracy even before threshold is reached.

The main objective of this paper to give a procedure that yields an upper and lower bound on the variance of registration error for a registration system that is operating under conditions where the registration error is constrained to be less than some maximum. The procedure is applied to a fairly general, one dimensional, discrete model, where the cross-correlation function is used as the matching function and where signal and noise power spectra are known. Results are presented for signals with a variety of spectra.

The organization of the paper is as follows. Section 2 outlines the procedure used to obtain the upper and lower bounds for the variance of registration error. Section 3 describes the model to which the procedure is applied. Section 4 obtains an expression for the approximate variance of registration error, while section 5 contains the results of four case studies. Concluding comments are contained in section 6.

II. THE BOUNDING PROCEDURE

The procedure for finding a lower (upper) bound on the variance of registration error involves finding a new mathematically attractive matching function that yields a smaller (larger) registration error for each and every registration. Then based on the new matching function the variance of registration error is calculated. This calculated variance is a lower (upper) bound on the variance of registration error for the system of interest.

In the past many have used the Taylor series approach to obtain an approximate variance of registration error for a variety of registration models[1][2][4][5][6]. The same approach can be used to obtain lower and upper bounds on the variance of registration error.

The Taylor series approach involves approximating the matching function in a way that makes the calculation of the variance of registration error possible. This is done by replacing the noise independent portion of the matching function with a parabolic approximation. The curvature of the parabola is chosen to give a good approximation to the matching function in the region near correct registration. The curvature of this parabola can, however, also be chosen so that the registration error incurred is less (more) than the actual registration error for each and every registration. The variance of registration error based on the parabolic approximation can be calculated and forms a lower (upper) bound on the variance of error for the model of interest.

The difficulty is finding the curvature that ensures the approximated matching function yields a smaller (larger) registration error for each and every registration. In pursuit of this curvature the following proposition is useful.

Proposition 1

A matching function m_1 yields an equal or smaller registration error than matching function m_2 for each and every registration providing:

1) m_1 and m_2 can be expressed as the sum of two components, a signal component and a noise component.
2) The signal component for both m_1 and m_2 is an even function about the correct registration position.
3) The noise component of m_1 is the same as that of m_2.
4) The slope (with respect to registration shift) of the signal component of m_1 is less than or equal to the slope of the signal component of m_2 for registration shifts greater than the shift at correct registration. (By condition 2 the slopes of the signal components are odd about the correct registration position. Therefore, for registration shifts less than the shift at correct registration the converse is true, the slope of m_1 is greater than or equal to the slope of m_2.)

Proof

The proof of this proposition will be done by contradiction. It will be assumed that the registration error incurred using m_1 is greater than that incurred using m_2. This leads to a mathematical contradiction which implies the assumption is false.

Let X_1 and X_2 be the registration shifts that maximize m_1 and m_2 respectively for a particular registration. Let X_0 be the shift for correct registration. Assume the registration error incurred using m_1 is greater than that incurred using m_2. Consider the four cases a) $X_1 \geq X_0$, $X_2 \geq X_0$; b) $X_1 < X_0$, $X_2 \geq X_0$; c) $X_1 < X_0$, $X_2 < X_0$; and d) $X_1 \geq X_0$, $X_2 < X_0$ which cover all possible values for X_1 and X_2.

Proof for case a)

The proof starts by assuming the registration error for matching function m_1 is greater than that for m_2; that is $X_1 > X_2$. By definition

$$m_1(X_1)-m_1(X_2) = \int_{X_2}^{X_1} \frac{dm_1(x)}{dx}\,dx, \text{ and} \tag{p1}$$

$$m_2(X_1)-m_2(X_2) = \int_{X_2}^{X_1} \frac{dm_2(x)}{dx}\,dx. \tag{p2}$$

By conditions 3) and 4) of the proposition the integrand of the right side of (p1) is less than or equal to the integrand of the right side of (p2) for all x in the interval (X_2,X_1). This implies $m_1(X_1)-m_1(X_2) \leq m_2(X_1)-m_2(X_2)$. Since X_1 maximizes m_1 and X_2 maximizes m_2 the above inequality has a positive number less than or equal to a negative number. Therefore the assumption that X_1 is greater than X_2 must be false and the proof for case a) is complete.

Proof for case b)

The proof starts by assuming the registration error for matching function m_1 is greater than that for m_2; that is $-(X_1-X_0) > (X_2-X_0)$. By definition

$$m_1(X_2)-m_1(X_1) = \int_{X_1}^{X_0-(X_2-X_0)} \frac{dm_1(x)}{dx} dx + \int_{X_0-(X_2-X_0)}^{X_0+(X_2-X_0)} \frac{dm_1(x)}{dx} dx, \text{ and} \quad \text{(p3)}$$

$$m_2(X_2)-m_2(X_1) = \int_{X_1}^{X_0-(X_2-X_0)} \frac{dm_2(x)}{dx} dx + \int_{X_0-(X_2-X_0)}^{X_0+(X_2-X_0)} \frac{dm_2(x)}{dx} dx. \quad \text{(p4)}$$

By conditions 2) the slope of the signal components are odd about X_0 and by condition 4) the noise components are identical which makes the right most integrals of (p3) and (p4) equal. By condition 3) and 4) the remaining integral of (p3) is greater than or equal to that of (p4). This implies $m_1(X_2)-m_1(X_1) \geq m_2(X_2)-m_2(X_1)$. Since X_1 maximizes m_1 and X_2 maximizes m_2 the above inequality has a negative number greater than or equal to a positive number. The assumption that $-(X_1-X_0)>(X_2-X_0)$ is therefore false and the proof of case b) is complete.

The proofs of cases c) and d) are similar to those of a) and b) and therefore omitted. This completes the proof of the proposition.

The procedure for finding the lower (upper) bound on the variance of registration error can now be completed using proposition 1. The procedure is as follows:

1) The matching function of interest is to satisfy the conditions on m_2 (m_1) of proposition 1.
2) A new matching function, based on approximating the matching function of interest, is created to satisfy the conditions on m_1 (m_2) of proposition 1. This is done by replacing the signal component of the matching function of interest with a parabola. The curvature of the parabola is chosen to satisfy condition 4) of proposition 1.
3) The variance of registration error is calculated for the approximation based matching function. This variance is a lower (upper) bound on the variance of registration error for the matching function of interest.

When the procedure is used as stated the upper bound on the variance of registration error will in general be infinity. This is the case since the parabolic signal component of the approximation based matching function will, in general, have to be concave up to satisfy condition 4) of proposition 1. The approximation based matching function will then be maximum at infinity yielding infinite registration error on each and every registration.

A meaningful upper bound is gained with knowledge of the maximum possible registration error. In most practical systems registration errors are usually constrained to be less than some known maximum. For registration systems that permit large registration errors have not accomplished their purpose and are of little use. So for most registration systems condition 4) of proposition 1 need only apply for the interval of registration shifts that produce errors less than this maximum. In which case the procedure will most likely result in a meaningful upper bound.

A typical matching function has been decomposed into signal and noise components which are shown in Figure 1. For such a matching function the registration system would likely be operating at signal to noise ratios that confine the global maximum to the main lobe of the signal component. The parabolic signal components that yield upper and lower bounds on the variance of registration are also shown in Figure 1 for the case where the peak of the matching function of interest is known to occur in the interval (X_{T-}, X_{T+}).

The curvature of the parabolic signal components yielding a bound can be obtained quite easily from a graph of the slope (with respect to registration shift) of the signal component versus registration shift. Such a graph is illustrated in Figure 2. Note that a parabolic signal component has a slope proportional to registration shift and appears as a straight line. This line must cross zero at the shift of correct registration (denoted by δ in Figure 2) since the parabolic signal component is an even function about that shift. The curvature of the parabolic signal component is simply the slope of the line. If the line passes under (over) the slope of the signal component throughout the interval of interest, denoted (X_{T-}, X_{T+}), then the slope of the line is the curvature of a parabolic signal component that yields a smaller (larger) registration error. It is clear from proposition 1 that the curvature of the parabola yielding the lower (upper) bound should be as small (large) as possible to provide the greatest (least) lower (upper) bound.

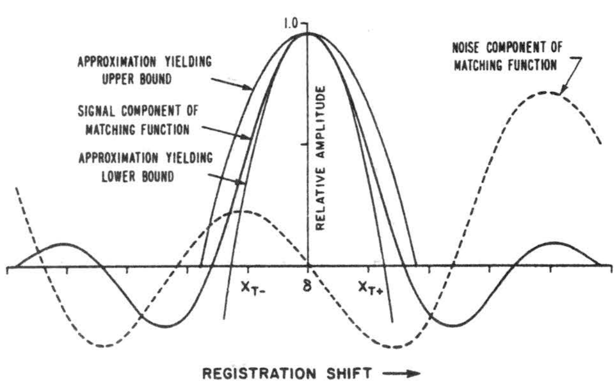

Figure 1. Parabolic Approximations to Matching Function.

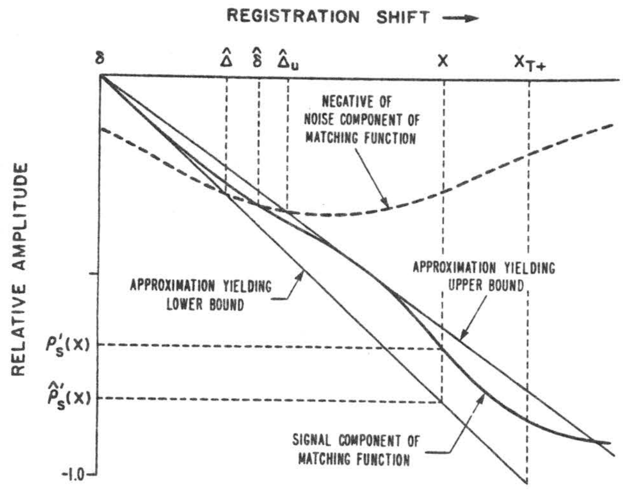

Figure 2. Matching Function Components and Approximations in the Derivative Space.

Some insight into the registration errors incurred by the approximated and the original matching functions can be gained by plotting the negative of the slope of the noise component in Figure 2. The maximum of the matching function occurs at the point where the slopes of the noise and signal components cross. For the particular noise component of Figure 2 the registration errors for the three matching functions illustrated are denoted $\hat{\Delta}$, $\hat{\delta}$, and $\hat{\Delta}_U$.

If it is assumed that the matching function has the same signal component for each and every registration, as is often the case, then the slopes of the lines indicated in Figure 2 are the curvatures of the parabolic signal components that will provide greatest lower and least upper bounds on the variance of registration error.

III. THE MODEL

A block diagram of the model to be analyzed is shown in Figure 3. At the heart of the system is the registration processor. This processor is presented with the sampled form of two signals referred to as the reference and registrant signals. The reference signal is a continuous analog signal consisting of a sample function $s(x)$, from a signal source, corrupted by a sample function $n_1(x)$, from an additive noise source. The registrant signal is also a continuous analog signal consisting of a shifted version of $s(x)$ (shift is δ), corrupted by sample function $n_2(x)$, from an additive noise source. The objective of the registration processor is to take the sampled form of the reference and registrant signals and obtain an estimate of the shift or misregistration δ on a sample function by sample function basis. Both the estimated misregistration, $\hat{\delta}$, and the registration error, $(\hat{\delta}-\delta)$, are random variables. The variance of the registration error is taken as a indicator of the performance of the registration processor and is therefore the prime quantity of interest.

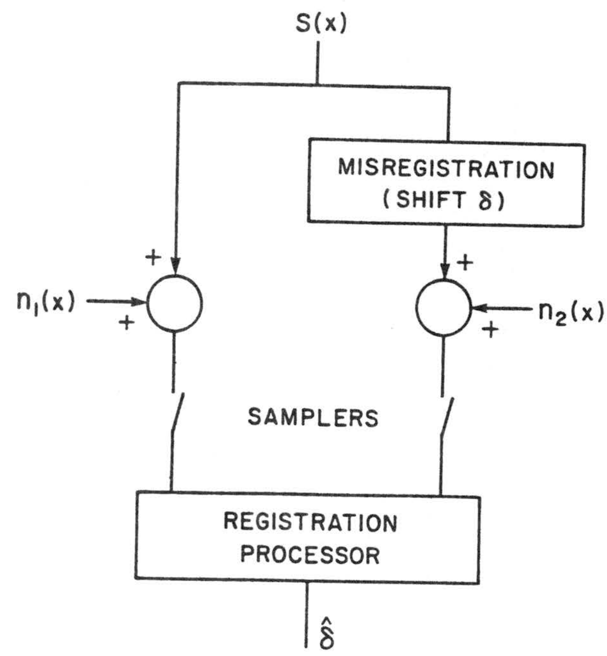

Figure 3. Registration System Model

In analyzing the registration model the following assumptions are made. The analog signal $s(x)$ is assumed to be a sample function from an ergodic, band limited (low-pass) periodic stochastic process with known power spectrum. The analog noise $n_1(x)$ and $n_2(x)$ are assumed to be sample functions from a stationary, band limited (low-pass), periodic, stochastic process with known power spectra. It is assumed that the signal and noise are periodic with identical periods and in addition the signal and both noise sources are assumed independent.

In sampling the reference and registrant signals it is assumed that the samples are taken simultaneously at a rate harmonically related to the common period of the signals being sampled. Consequently, exactly an integer number of samples, M, are collected during each period of the signals. As is common in discrete analysis the sampling interval is assumed to be one unit of x without loss of generality. The sampling rate is assumed to be sufficiently rapid so that the Nyquist criterion for the low-pass signals being sampled is satisfied.

As previously mentioned the registration processor estimates the misregistration δ. This estimate is determined by cross-correlating the continuous reference and registrant signals and taking the value of shift which achieves a maximum value for the cross-correlation function as $\hat{\delta}$ the estimate for δ (i.e. cross-correlation is used as the matching function). This means that the estimated misregistration $\hat{\delta}$ can assume a continuum of values as can the registration error $(\hat{\delta}-\delta)$. The ability of the processor to obtain a continuum of estimates for δ from discrete samples of the reference and registrant signals stems from the assumptions made regarding signals and sampling rate, which eliminate aliasing and permit an exact analytical expression for the signals in terms of their samples.

A comment appears in order regarding the assumption that the signal as well as the noise sample functions are periodic. In practical signal matching applications signals must ordinarily be truncated, resulting in edge effects for such operations as correlation. Assuming that the signals are periodic is simply one method of handling these edge effects and appears as valid as other commonly used methods such as padding the functions with zeros. The assumption that the signals are periodic has the distinct advantage of greatly simplifying the theoretical analysis, primarily because it fits well with the natural periodicities generated by discrete transform techniques.

Some comments regarding the generality of the model in Figure 3 conclude this section. Since signal and noise are characterized by their power spectra linear operations which simply modify these spectra are readily included in the model. In addition the model can accommodate determination of variance of registration error for "image like" signals which change with time. This is accomplished by setting $n_1(x)$ to zero and letting $n_2(x)$ account for the change in the "image like" signals. Thus the model in Figure 3, which at first glance appears quite restrictive, can in fact handle a variety of situations.

IV. APPROXIMATION BASED VARIANCE

This section contains the derivation of an expression for the variance of registration error for a matching function based on a parabolic signal component. The resulting expression will be of use in calculating the lower and upper bounds on the variance of registration error for the registration system model of Figure 3.

First the matching function of the registration processor, which is the cross-correlation function, is analyzed and expressed in terms of the sampled reference and registrant signals. The matching function is then broken into a signal and noise component. The signal component is approximated with a parabola and an expression for the variance of registration error is found using this approximation.

A. MATCHING FUNCTION

The matching function, which is the cross-correlation function $\rho(x)$, used for a particular registration is a sample function from a stochastic process. Each sample function $\rho(x)$ is related to the sample functions of the signal and the noise by

$$\rho(x) = \int_0^M (s(y)+n_1(y))\,(s(y+x-\delta)+n_2(y+x))\,dy, \tag{1}$$

where M is the common period of the signal and the noise.

Since the signal and noise are low-pass periodic functions they can be expressed by a finite Fourier series expansion. Since they also satisfy the Nyquist critera the only possible non zero Fourier components are at frequencies $\frac{2\pi k}{M}$, $k=0,1,2,\cdots,\frac{M-1}{2}$ (Throughout this paper it is assumed M is an odd integer. The theory applies for M even providing all occurrences of $\frac{M-1}{2}$ are changed to $\frac{M}{2}-1$.) The integrand of (1) being the product of two such functions has possible non zero Fourier components at frequencies $\frac{2\pi k}{M}$, $k=0,1,2,\cdots,2\frac{M-1}{2}$. Using the identities

$$\int_0^M \cos(\frac{2\pi k}{M}y+\theta_k)dy = \begin{cases} M\cos\theta_k & k=0 \\ 0 & \text{otherwise and} \end{cases} \tag{2a}$$

$$\sum_{m=0}^{M} \cos(\frac{2\pi k}{M}m+\theta_k) = \begin{cases} M\cos\theta_k & k=0,\pm M,\pm 2M,\cdots \\ 0 & \text{otherwise,} \end{cases} \tag{2b}$$

where k and M are integers, (1) can be transformed into a mathematically discrete form given by

$$\rho(x) = \sum_{m=0}^{M-1} \Big(s(m)+n_1(m) \Big)\Big(s(m+x-\delta)+n_2(m+x) \Big)\,. \tag{3}$$

It must be emphasized that under the given assumptions the computationally convenient discrete form of (3) gives exactly the same values for $\rho(x)$ as the continuous form given by (1) for all values of the shift parameter x. This means $\rho(x)$ can be evaluated for the continuum of real x in terms of the sampled values of the reference and registrant signals. Therefore the value of x that maximizes $\rho(x)$ can be found exactly.

The matching function given in (3) is separated into a signal component $\rho_s(x)$ and noise component $\rho_n(x)$ such that

$$\rho(x) = \rho_s(x) + \rho_n(x)\,. \tag{4}$$

The signal component is given by

$$\rho_s(x) = \sum_{m=0}^{M-1} s(m)s(m+x-\delta) \tag{5}$$

and the noise component is given by

$$\rho_n(x) = \sum_{m=0}^{M-1}\Big\{ s(m)n_2(m+x) + n_1(m)s(m+x-\delta) + n_1(m)n_2(m+x)\Big\} . \qquad (6)$$

The signal component $\rho_s(x)$ is an even function about $x=\delta$. This is shown by expressing (5) as an integral using the same reasoning that equates the right sides of (1) and (3) and using the periodic properties of the integrand. Thus $\rho_s(x)$ satisfies condition 2) of proposition 1 and facilitates the use of the use of the previously described procedure to gain bounds on the variance of registration error.

B. VARIANCE OF APPROXIMATED MATCHING FUNCTION

In this section an expression is developed for the variance of registration error for an approximation based matching function. The approximated matching function, $\hat{\rho}(x)$, is given by

$$\hat{\rho}(x) = \hat{\rho}_s(x) + \rho_n(x), \qquad (7)$$

where $\hat{\rho}_s(x)$ is a parabolic approximation of the signal component $\rho_s(x)$. The parabola has curvature C and is given by

$$\hat{\rho}_s(x) = C(x-\delta)^2. \qquad (8)$$

If one assumes $\hat{\rho}(x)$ is unimodal, having one and only one maximum, then the location of its maximum can be found by setting the derivative of $\hat{\rho}(x)$ to zero and solving for x. Combining (7) and (8) and letting $\hat{\Delta}$ be the value of x which makes the derivative of $\hat{\rho}(x)$ zero results in the transcendental equation

$$\hat{\Delta} = \delta - \frac{\rho_n'(\hat{\Delta})}{2C}, \qquad (9)$$

where for notational convenience $\rho_n'(\hat{\Delta})$ is used to indicate $\left.\frac{d\rho_n(x)}{dx}\right|_{x=\hat{\Delta}}$. The use of a prime to designate a derivative will be used throughout this paper.

Assuming $\hat{\rho}(x)$ is unimodal, (9) will have a unique solution and $\hat{\Delta}$ is a legitimate random variable. The mean and variance of the registration error, $(\hat{\Delta}-\delta)$, can therefore be calculated. Evaluating the derivative of (6) at $x=\hat{\Delta}$ and substituting the result into (9) yields

$$\hat{\Delta} - \delta = -\frac{1}{2C}\sum_{m=0}^{M-1}\Big\{ s(m)\, n_2'(m+\hat{\Delta}) + n_1(m)\, s'(m+\hat{\Delta}-\delta) + n_1(m)\, n_2'(m+\hat{\Delta})\Big\}. \qquad (10)$$

Taking the expectation of (10) and interchanging the order of expectation and summation operators gives

$$E[\hat{\Delta} - \delta] = -\frac{1}{2C}\sum_{m=0}^{M-1}\Big\{ E[s(m)\, n_2'(m+\hat{\Delta})] + E[n_1(m)\, s'(m+\hat{\Delta}-\delta)] + E[n_1(m)\, n_2'(m+\hat{\Delta})]\Big\}. \qquad (11)$$

It is believed that independence of the signal and noise holds for the transcendental equation of (11) even thought it has not been formally proven*. Using independence, the expectation of the products on the right side of (11) can be written as the product of expectations. Each term in (11) is zero since one factor in each term is the expected value of a random variable defined on the derivative of a sample function from a stationary stochastic process and that factor must be zero. This establishes the mean of the registration error

$$E[\hat{\Delta} - \delta] = 0 . \qquad (12)$$

The variance of registration error is given by $E[(\hat{\Delta} - \delta)^2]$ since the mean of $\hat{\Delta}-\delta$ is zero. It is obtained by taking the expectation of the square of (10). The square of (10) produces an equation with nine terms under a double summation on its right side. Using the same procedure and reasoning as in the development of (11), the expected value of six of these terms is zero. The variance of error expression becomes

$$\sigma_{\hat{\Delta}}^2 = \frac{1}{4C^2}\sum_{h=0}^{M-1}\sum_{m=0}^{M-1}\Big\{ E[s(m)n_2'(m+\hat{\Delta})s(h)n_2'(h+\hat{\Delta})] + E[n_1(m)s'(m+\hat{\Delta}-\delta)n_1(h)s'(h+\hat{\Delta}-\delta)] + E[n_1(m)n_2'(m+\hat{\Delta})n_1(h)n_2'(h+\hat{\Delta})]\Big\}. \qquad (13)$$

In evaluating the expectations in (12) it is helpful to make a change of variables, letting $p=h-m$. The limits for the summation over the new variable, p, are independent of m since each term under the double summation in the new expression, formed by replacing h with $p+m$, is periodic in p with period M. This allows the limits for the summation over p to be fixed at 0 and $M-1$. Using independence of the signal and noise sources the expectation of the new expression is evaluated in terms of the autocorrelation functions of the processes that generate the signal and the noise. These function are only dependent on p permitting the summation over m to be carried out with the result

$$\sigma_{\hat{\Delta}}^2 = \frac{M}{4C^2}\sum_{p=0}^{M-1}\Big\{ -R_s(p)R_2''(p) - R_1(p)R_2''(p) - R_1(p)R_2''(p)\Big\}, \qquad (14)$$

* Others have made the same assumption in similar transcendental equations(1)(2)(4)(5)(6).

where $R_s(x)$, $R_1(x)$ and $R_2(x)$ are the autocorrelation functions of the processes generating $s(x)$, $n_1(x)$ and $n_2(x)$ respectively.

The variance of registration error can be expressed in terms of the power spectra of the signal and the noise. Since the processes generating the signal and the noise are periodic with period M their power spectra are a sequence of impulses spaced by $\frac{2\pi}{M}$. Let $S(k)$, $N_1(k)$ and $N_2(k)$ be the power at frequency $\frac{2\pi k}{M}$, $k=0,1,2,\cdots,\frac{M-1}{2}$, in the processes generating the signal, reference noise and registrant noise respectively. $S(k)$, $N_1(k)$ and $N_2(k)$ are then given by the kth coefficient of the cosine Fourier series of the respective autocorrelation functions. Replacing the autocorrelation functions in (14) with their cosine Fourier series representation produces a triple summation containing the $S(k)$, $N_1(k)$ and $N_2(k)$. Summing over p and using (2b) reduces the triple summation to a single summation. The expression for the variance of registration error becomes

$$\sigma_{\hat{\Delta}}^2 = \frac{M}{8C^2}\sum_{k=0}^{\frac{M-1}{2}}\left[\frac{2\pi k}{M}\right]^2\Big\{S(k)N_1(k) + S(k)N_2(k) + N_1(k)N_2(k)\Big\}. \tag{15}$$

For the special case where the signal and noise have the same spectral shapes (15) can be rearranged to yield

$$\sigma_{\hat{\Delta}}^2 = \frac{1}{8C^2}\left[\frac{2\,SNR+1}{SNR^2}\right]\sum_{k=0}^{\frac{M-1}{2}}\left[\frac{2\pi k}{M}\right]^2 S(k)^2, \tag{16}$$

where SNR is the signal power to noise power ratio of the reference and registrant signals.

V. BOUNDS ON VARIANCE

Both lower and upper bounds on the variance of registration error can be calculated using (15). All that is required is the curvatures of the parabolic signal components that yield the lower and upper bounds.

The lower bound is found first. The approximation based matching function yielding the lower bound must satisfy the conditions on m_1 of proposition 1 while the matching function of interest must satisfy the conditions on m_2 of proposition 1. Conditions 1), 2) and 3) are satisfied for any curvature. Condition 4) is satisfied providing the slopes of the signal components satisfy the inequalities

$$\hat{\rho}_s'(x) \le \rho_s'(x),\quad \delta \le x \le X_{T+} \text{ and} \tag{17a}$$

$$\hat{\rho}_s'(x) \ge \rho_s'(x),\quad X_{T-} \le x < \delta, \tag{17b}$$

where it is known that the peak of the matching function of interest is maximum in the interval (X_{T-},X_{T+}).

In finding the curvature of the parabolic signal component that satisfies (17a) and (17b) it is helpful to establish that the slope of $\rho_s'(x)$ is minimum at $x=\delta$. Consider the expression for $\rho_s(x)$ given by (5). The right side of (5) can be expressed in terms of the power spectrum of the signal. This is done by replacing $s(m)$ and $s(m+x-\delta)$ with their cosine (amplitude and phase) Fourier series representation which results in a triple summation. Since the signal is ergodic the square of the Fourier amplitude coefficient for frequency $\frac{2\pi k}{M}$ is equal to $2S(k)$. Using trigonometric identities, summing over m, and using (2b) reduces the triple summation to a single summation given by

$$\rho_s(x) = M\sum_{k=0}^{\frac{M-1}{2}} S(k)\cos\left(\frac{2\pi k}{M}(x-\delta)\right). \tag{18}$$

Taking the derivative of (18) twice yields

$$\rho_s''(x) = -M\sum_{k=0}^{\frac{M-1}{2}}\left[\frac{2\pi k}{M}\right]^2 S(k)\cos\left(\frac{2\pi k}{M}(x-\delta)\right), \tag{19}$$

where $\rho_s''(x)$ is the slope of $\rho_s'(x)$. Since $S(k)$ is greater than or equal to zero for all k it is clear from (19) that $x=\delta$ minimizes $\rho_s''(x)$.

To satisfy (17a) and (17b) in the vicinity of correct registration shift (i.e. x near δ) the slope of $\hat{\rho}_s'(x)$ must be less than or equal to $\rho_s''(\delta)$. As it was established $\rho_s''(x)$ is minimum at $x=\delta$, if $\hat{\rho}_s'(x)$, which is a straight line crossing zero at $x=\delta$, has slope $\rho_s''(\delta)$ then (17a) and (17b) are satisfied for all x. Since $\hat{\rho}_s'(x)$ could not have a greater slope and still satisfy (17a) and (17b) for x near δ, $\rho_s''(\delta)$ is the slope of $\hat{\rho}_s'(x)$ that yields the greatest lower bound.

The curvature that yields the greatest lower bound, C_L, is found by taking the derivative of (8) twice and substituting $\rho_s''(\delta)$ for $\hat{\rho}_s''(x)$ and is given by

$$C_L = \frac{\rho_s''(\delta)}{2}. \tag{20}$$

Evaluation of (19) at $x=\delta$ and substitution into (20) yields an expression for the curvature in terms of the power spectrum of the signal given by

$$C_L = -\frac{M}{2}\sum_{k=0}^{\frac{M-1}{2}}\left[\frac{2\pi k}{M}\right]^2 S(k). \tag{21}$$

It is pointed out that C_L does not depend on the registration error being constrained to less than some maximum. It is also pointed out that curvature C_L is the coefficient for the second term in the Taylor series expansion of $\rho_s(x)$ about $x=\delta$. The matching function used to calculate the lower bound is the same matching function yielded by the Taylor series approach. The lower bound should therefore approach the variance of registration error for the matching function of interest when the registration errors are small.

An upper bound on the variance of registration error can be expressed in terms of a lower bound on the variance of registration error. From (15) it is observed that the variance of registration error for a parabolic signal component is inversely proportional to the curvature squared. An upper bound on the variance of registration error is therefore given by

$$\sigma_U^2 = \sigma_L^2 \frac{C_U^2}{C_L^2}, \tag{22}$$

where σ_U^2 and σ_L^2 are an upper bound and a lower bound on the variance of registration error and C_U and C_L are the curvatures of the parabolic signal components used to obtain σ_U^2 and σ_L^2 respectively.

Consider the ratio of the slopes of the parabolic signal components with curvatures C_U and C_L. Using the derivative of (8) this ratio is given by

$$\frac{2\,C_U(x-\delta)}{2\,C_L(x-\delta)} = \frac{C_U}{C_L}. \tag{23}$$

From condition 4) of proposition 1 for $x>\delta$ the slope of the parabolic signal component yielding the upper bound must be greater than or equal to the slope of the signal component of the matching function of interest. The least upper bound on the variance of registration error is achieved if the two slopes are equal for some possible x. The ratio $\frac{C_U}{C_L}$ where C_U is the curvature yielding a least upper bound is therefore given by

$$\frac{C_U}{C_L} = \max_x \frac{\rho_s'(x)}{\hat{\rho}_s'(x)}, \quad X_{T-} \leq x \leq X_{T+}, \tag{24}$$

where $\hat{\rho}_s'(x)$ is the slope of the parabolic signal component that has curvature C_L and where the peak of the matching function of interest is known to occur in the interval (X_{T-},X_{T+}).

The curvature ratio can be expressed in terms of the power spectrum of the signal. Using the derivative of (18), derivative of (8) and (21) yields

$$\frac{C_U}{C_L} = \max_x \frac{\sum_{k=0}^{\frac{M-1}{2}} \left[\frac{2\pi k}{M}\right] S(k)\sin\left(\frac{2\pi k(x-\delta)}{M}\right)}{(x-\delta)\sum_{k=0}^{\frac{M-1}{2}} \left[\frac{2\pi k}{M}\right]^2 S(k)},$$

$$X_{T-} \leq x \leq X_{T+}. \tag{25}$$

VI. EXPERIMENTAL RESULTS

The lower and upper bounds on the variance of registration error are plotted in Figures 4 and 5 for a variety of signals registered according to the model in Figure 3. In each case the stochastic processes generating the signal and the noise have the same power spectra. The domain of the signal to noise ratios used in Figures 4 and 5 was restricted so that the peak in each matching function is sure to be on the main lobe of the signal component between the points X_{T-} and X_{T+} where X_{T-} and X_{T+} are the two solutions in x to $\rho_s(x)=\frac{\rho_s(\delta)}{2}$.

The correlation interval M was chosen 128. This yields power spectra with mass points at frequencies $\frac{\pi k}{64}$ for $k=0,1,2,\cdots,63$. Four signal power spectra are used:

1) Low-pass

$$S(k) = \begin{cases} 0 & k=0 \\ 1 & 1 \leq k \leq 15 \\ 0 & 16 \leq k \leq 63 \end{cases}$$

2) Band-pass

$$S(k) = \begin{cases} 0 & 0 \leq k \leq 23 \\ 1 & 24 \leq k \leq 39 \\ 0 & 40 \leq k \leq 63 \end{cases}$$

3) Triangular

$$S(k) = \begin{cases} 0 & k=0 \\ 1-\frac{k}{63} & 1 \leq k \leq 63 \end{cases}$$

4) Contrived

$$S(k) = \begin{cases} 0 & k=0 \\ 1 & 1 \leq k \leq 15 \\ 0 & 16 \leq k \leq 62 \\ 1.5 & k=63 \end{cases}$$

The first three spectra represent a variety of situations that could well be encountered in practice. Figure 4 contains graphs for these cases. The lower bound is plotted using (16) and the upper bounds using (22). The fourth spectrum was contrived to make the upper bound on the variance of registration error infinite. The lower bound for this contrived case has been plotted in Figure 5 using (16).

The validity of the bounds is tested using simulation based estimates of the actual variance of registration error for several signal to noise ratios. Each variance estimate was calculated using results of 25,000

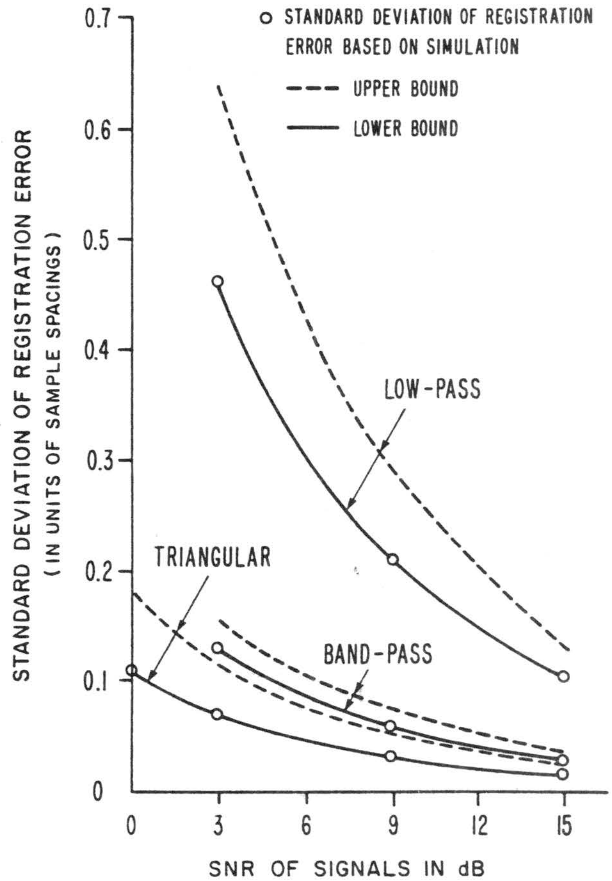

Figure 4. Bounds on Standard Deviation of Registration Error

registrations. The estimates are shown in Figures 4 and 5 as small circles.

It is pointed out that the variance of registration error used for the lower bound is also the variance of error obtained using the Taylor series approach to best approximate the matching function of interest. It is therefore not surprising that the experimentally obtained points fall very near the lower bound at small signal to noise ratios.

Some comments regarding the computational efficiency of the bounding procedure are in order. A VAX 11/780 digital computer was used. The simulation based data required .26 seconds for each registration and 6500 (25000×0.26) seconds for each estimate shown in Figures 4 and 5. An upper and lower bound pair of points were obtained in .07 seconds.

It is also interesting to note that the number of registrations required in one simulation for the standard deviation of the estimate to be one quarter the difference between the upper and lower bounds is about 64, 200 and 800 registrations for the triangular, low-pass and band-pass spectra respectively.

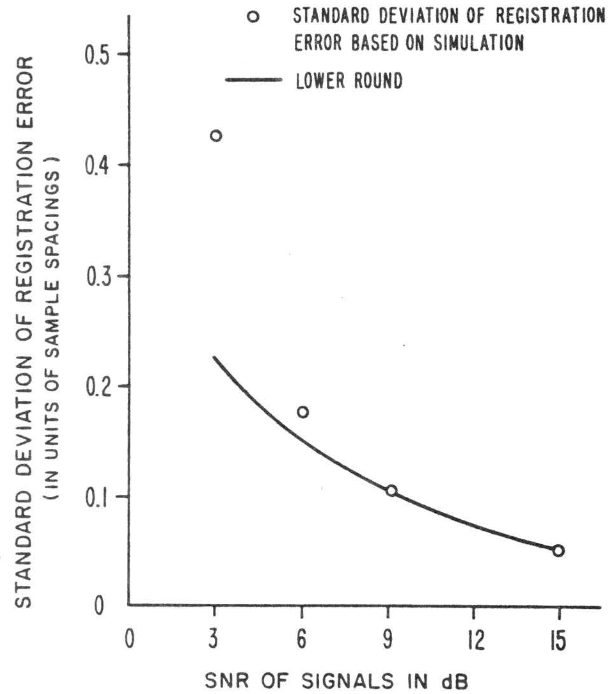

Figure 5. Lower Bound on Standard Deviation of Registration Error for Contrived Spectrum.

VII. CONCLUDING SUMMARY

Upper and lower bounds on the variance of registration error can be calculated for registration systems where small error analysis techniques are used to obtain an approximate variance of registration error. These bounds are useful in establishing the accuracy of the approximate variance.

The closeness of the upper and lower bound is in general affected by the maximum possible registration error. The bounds approach each other as the maximum possible registration error decreases.

Bounds were calculated and tested against simulation based data for the registration system model of Figure 3, which uses a cross-correlator as a matching function. The lower bound for a cross-correlator matching function was shown to be independent of the range of possible registration errors.

The upper and lower bounds were found to be meaningful for a variety of practical power spectra. However, a spectra was contrive to yield a meaningless upper bound. Even thought there are situations where the upper bound is not meaningful the results indicate that for most practical situations the upper and lower bounds will be useful.

VIII. ACKNOWLEDGEMENTS

The support for this work was provided by the Natural Sciences and Engineering Research Council of Canada and the University of Saskatchewan.

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