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# Notes on Image Correlation

# and

# Registration System Improvements

by

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## Introduction

Investigation has been conducted into means of improving the image correlator and overlay function portions of the LARS image registration system. The fast Fourier transform correlator was analysed in detail and a modified version was programmed and tested. More reliable and accurate correlation has been achieved on both multispectral and multitemporal scanner imagery and Apollo 9 digitized multispectral photography. A quadratic overlay function was developed which uses local correlation checkpoints to interpolate the required overlay shifts between checkpoints. These improvements are described in this memo. The modified correlator is checked out and ready for inclusion in the registration system. More work is required on the overlay function to implement it in the system.

## Philosophy Behind the "Improved" Correlation Scheme

The "improved" correlation algorithm doesn't significantly differ from the existing scheme. The major deviation lies in the fact that all data in both the two N x N arrays are used, instead of the previous N x N array being correlated with a N/2 x N/2 array. Due to the cyclic properties of the two dimensional FFT scheme, the "improved" algorithm is not the "theoretically correct" scheme in the same sense that the Sequential Similarity Detection Algorithm (SSDA) devised by IBM (RG-3356) is not in that the values of correlation near the peak (small shift-values of the two functions) will be "nearly correct", and no error occurs at the center of the 2-D correlation function. In general, correlation values on either side of the center are useful up to a "delta" of N/4. If N=64, a deviation of 16 pixel points either way can usually be determined through the use of this "improved" scheme.

## Symmetry Properties of a 2-Dimensional Real Matrix

The major symmetry characteristics of the Fast 2-D Fourier Transform of a N x N real matrix is listed, where n = 2**m. Any complex matrix whose real and imaginary parts satisfy these symmetry conditions will be inverse Fourier Transformable to a <u>real matrix</u>. Any 2-D correlation function must be real, thus its fourier transform must possess the following properties:

(1) Real part of the transform $\{a_{ij}\}$:

$$
\begin{array}{ccccc}
a_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots\cdots & a_{2n} \\
\vdots & & & & \vdots \\
a_{\frac{n}{2}+1,1} & a_{\frac{n}{2}+1,2} & & \cdots\cdots & a_{\frac{n}{2}+1,n} \\
\vdots & & & & \vdots \\
a_{n1} & a_{n2} & & \cdots\cdots & a_{nn}
\end{array}
$$

Except for row $a_{\frac{n}{2}+1,l}$ $(l = 1,\ldots n)$, the submatrix outlined by the red has the symmetry:

$$
\begin{array}{llll}
a_{22} = a_{nn} & a_{23} = a_{n,n-1} & \cdots\cdots & a_{2n} = a_{n2} \\
a_{32} = a_{n-1,n} & a_{33} = a_{n-1,n-1} & \cdots\cdots & a_{3n} = a_{n-1,2} \\
a_{42} = a_{n-2,n} & a_{43} = a_{n-2,n-1} & \cdots\cdots & a_{4n} = a_{n-2,2}
\end{array}
$$

etc.

Three exceptions:

(a) Row one, $a_{1l}$, we have $a_{12} = a_{1n}$, $a_{13} = a_{1,n-1}$, $a_{14} = a_{1,n-2}\cdots$

(b) Row $\frac{n}{2}+1$, we have $a_{\frac{n}{2}+1,2} = a_{\frac{n}{2}+1,n}$, $a_{\frac{n}{2}+1} = a_{\frac{n+1}{2},n-1}$, $a_{\frac{n}{2}+1,4} = a_{\frac{n}{2}+1,n-2}\ldots$

(c) Column one, $a_{l1}$, $a_{12} = a_{n1}$, $a_{13} = a_{n-1,1}$, $a_{14} = a_{n-2,1}\cdots$

(2) Imaginary part of the transform $\{b_{ij}\}$:

$$\begin{array}{ll}
b_{11} & \begin{matrix} b_{12} & b_{13} & b_{14} \cdots\cdots & b_{1,n-1} & b_{1n} \end{matrix} \\
\begin{matrix} b_{12} \\ \\ \\ \\ \\ b_{\frac{n}{2}+1,1} \\ b_{n1} \end{matrix} &
\begin{bmatrix}
b_{22} & b_{23} & \cdots\cdots\cdots & & b_{2n} \\
b_{32} & b_{33} & \cdots\cdots\cdots & & b_{3n} \\
\vdots & & & & \\
b_{\frac{n}{2}+1,2} & & & & b_{\frac{n}{2}+1,n} \\
b_{n2} & & & b_{n,n-1} & b_{nn}
\end{bmatrix}
\end{array}$$

Except for row $b_{\frac{n}{2}+1,l}$ $(l=1,\ldots n)$, the submatrix outlined by red has the symmetry:

$b_{11} = 0$

$b_{22} = -b_{nn}$ $\quad b_{23} = -b_{n,n-1}$ $\cdots\cdots$ $b_{2n} = -b_{n2}$

$b_{42} = -b_{n-2,n}$ $\quad b_{43} = -b_{n-2,n-1}\cdots\cdots$ $b_{4n} = -b_{n-2,2}$

Three exceptions:

(a) Row one, $b_{1l}$, $b_{12} = -b_{1n}$, $b_{13} = -b_{1,n-1}$, $b_{14} = -b_{1,n-2}\cdots$

(b) Row $(\frac{n}{2}+1)$, $b_{\frac{n}{2}+1,2} = -b_{\frac{n}{2}+1,n}$, $b_{\frac{n}{2}+1,3} = -b_{\frac{n}{2}+1,n-1}\cdots\cdots$

(c) Column one, $b_{l1}$, $b_{12} = -b_{n1}$, $b_{13} = -b_{n-1,1}$, $b_{14} = -b_{n-2,1}\cdots$

Derivation of the expressions used for computing the correlation function of two real one-dimensional functions by calling the Fast Fourier Transform only once and the Inverse Fast Fourier Transform also once are presented here. The expression to be derived below makes use of the so-called core-storage saving method for computing the correlation function of the two one-dimensional real functions. It makes use of the cyclic properties of the Fast Fourier Transform Algorithm such that before the transformation is taken, one function goes into the real part of the "Fourier Array" and the other function goes into the imaginary part of the array. In so doing, only one Fourier Transform operation is needed. The frequency components of the two functions are manipulated in the frequency domain according to some simple expressions so that upon taking the Inverse Fast Fourier Transformation, the correlation function results which turns out also to be real as expected. The correlation computation of two real two-dimensional functions followes essentially the same procedure, with only minor corrections due to the more complicated symmetry properties of the two-dimensional Fast Fourier Transform of a two-dimensional real function.

Given f(t), g(t), two real functions, we want to find the correlation

$$c(\tau) = \int_t f(t)\,g(t+\tau)\,dt$$

Let: $\mathcal{F}[f(t)] = F(w) = R(w) + i\,P(w)$, then $R(w) = R(-w)$, $P(w) = -P(-w)$

$F[g(t)] = G(w) = S(w) + i\,Q(w)$, then $S(w) = S(-w)$, $Q(w) = -Q(-w)$

and

$F[c(\tau)] = F(w).\bar{G}(w)$

Now let:

$a(t) = f(t) + i\,g(t)$

$A(w) = F[a(t)] = F[f(t)] + i\,F[g(t)] = [R(w) - Q(w)] + i[P(w) + S(w)]$

and

$A(-w) = [R(-w) - Q(-w)] + i[P(-w) + S(-w)] = [R(w) + Q(w)] + i[-P(w) + S(w)]$

$\bar{A}(-w) = [R(w) + Q(w)] - i[P(w) + S(w)]$

Now:

$A(w) + \bar{A}(-w) = 2\,R(w) + 2iP(w) = 2\,F(w)$

$A(w) - \bar{A}(-w) = -2[Q(w) - iS(w)] = 2i[S(w) + iQ(w)]$

and

$$F(w) = \frac{A(w) + A(w)}{2} \qquad G(w) = \frac{A(w) - \bar{A}(-w)}{2i}$$

From the cyclic properties of the Fast Fourier transform (See, e.g. "The Finite Fourier Transform" by J.W. Cooley, P. A.W. Lewis, P.D. Welch. IEEE transactions Vol. AV-17, No. 2, June 1969, pp.77]. We found that $\bar{A}(-n)=A(N-n)$ thus transforming $a(t)$ using Fast Fourier transform. We place continuous variable "W" by discrete variable "N" (N = 1.....N) we have

$$F(n) = R(n) + iP(n) = \frac{A(n) + A(N-n)}{2}$$

$$G(n) = F(n) \times G(n) = \underbrace{\left[\frac{A(n) + A(N-n)}{2}\right]\left[\frac{A(n)-A(N-n)}{2}\right]}_{X(n)+iY(n)} =$$

$$= \frac{1}{4i}\left[A(n)\bar{A}(n) + \bar{A}(N-n)\bar{A}(n) - A(n)A(N-n) - \bar{A}(N-n)A(N-n)\right]$$

Let: $A(n) = \alpha_n + iB_n \quad A(N-n) = \alpha_{N-n} + iB_{N-n}$

$$A(n) = \frac{1}{4i}[2(\alpha_n B_{N-n})^2 + (B_n - i\alpha_{N-n})^2]$$

$$= \frac{1}{4}[2(\alpha_n B_{N-n} + B_n \alpha_{N-n}) + i(B_n^2 - B_{N-n}^2 + \alpha_n^2 - \alpha_{N-n}^2)]$$

Real part of $A(n); X(n) = -\frac{1}{2}(\alpha_n \cdot \beta_n + \alpha_{N-n} \cdot \beta_n)$
and Imaginary part of
$A(n); Y(n) = -\frac{1}{4}(\alpha_n^2 + B_n^2 - \alpha^2_{N-n} - B^2_{N-n})$

Note if:

$B(n) = \bar{F}(n) \cdot G(n)$, then $B(n) = U(n) + iV(n)$ with $U(n) = \frac{1}{2}(\alpha_n B_{N-n} + B_n \alpha_{N-n})$

$V(n) = \frac{1}{4}(\alpha_n^2 + B_n^2 - \alpha^2_{N-n} - B^2_{N-n})$

## A Non-linear Localized Image Registration Approach

The concept of precise image overlay with two or more apertures using a localized non-linear programming approach based on a grid of "correct" correlation values between the reference and overlay apertures was conceived on December 21, 1971. Work has been carried on for three successive weeks until some programming problems were encountered when attempting to use actual data. However, some pilot programs were run to test out the concept and satisfactory results were obtained. It is at present difficult to say what modifications in the concept will have to be made in order for it to work satisfactorily under actual overlay data conditions, and work is to be continued along this line.

While testing out the pilot programs it has already been found that numerical precisions of the computer do contribute significantly to the error involved in the outcome, and care must be taken to avoid any large computational error so obtained. The source of this error camedirectly as a consequence of the least-square approach used. Work will be continued to get around this touchy point.

## Motivation:

The motivation of using a localized non-linear instead of the piece-wise linear approach in image registration consist of the following:

(1) The actual geometrical distortion of an aperature often is non-linear. This non-linear variation in a fairly restricted (localized) region can often by approximated by a lower (2 or 3) order two-dimensional polynomial or other suitable basis function that involves the computation of reasonable coefficients.

(2) The approximating functions for the geometrical distortion should be continuous from one region to the other. Not only should the approximating functions be continuous, their directional derivations at certain "checkpoints" should also be continuous to insure a very smooth transition from one region to the other.

(2) When the geometrical distortions over certain regions are very small or are fairly linear, the approach should be such that a larger region of the aperature can be corrected all at once. In other words, we will be desireable if the amount of computation involved in precise image registration can be made to proportion to the "severity" of the distortion of the overlay aperature relative to the reference aperature.

## Theory:

With the above three criteria in mind, the following approach making use of the non-linear programming technique with equality constraints imposed is derived. Consider a one-dimensional problem the correlation results are shown in the following graph (Fig. 1).

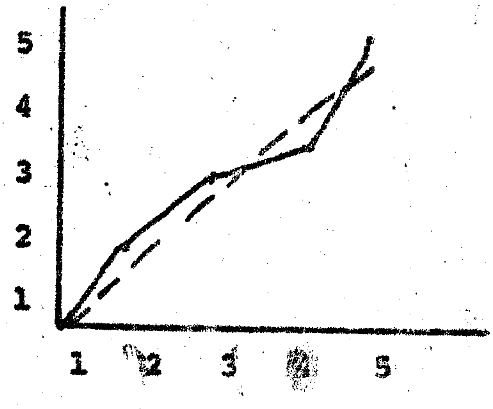


Figure 1

Instead of using the piece-wise linear approach, the curved line seemed to be much smoother and thus desiiebble.

Shown in Figure 2 is a correlation grid. The correlation values at each grid point (x) is computed. For example, at point A, we obtain (2,1). At point B, (2,1) again, but at point C, (2,2).

Based on the correlation values at the grid points (notice that the grid points do not need to be regularly spaced), we postulate a two-dimensional approximating function for the row distortion over a small region A-B-D-E and another two-dimensional function for the column distortion over the same region. The coefficients of these functions are determined not only from the four grid points A, B, D, E, but A-B-D-E to the next block B-E-C-F or the block D-E-G-H. In addition, the continuity at point A and the continuity of the x and y directional derivative at point A is assured by constraining equations when computing the approximating functions over block A-B-D-E. The same things hold true at point B when computing approximating functions over the block B-C-E-F.

Let g(x'y) be the approximating Set over A-B-D-E
Given:

$$g(x,y)|_A = Z \quad \frac{\delta g}{\delta x} = P, \ \frac{\delta g}{\delta y} = Q$$

we want to minimize

$$J = ||g_B - \underline{B}||^2 + ||g_D - D||^2 + ||g_G - \underline{G}||^2$$

subject to the condition

$$g_A = \underline{A} \quad \frac{\delta g}{\delta x} = P \quad \frac{\delta g}{\delta y} = Q$$

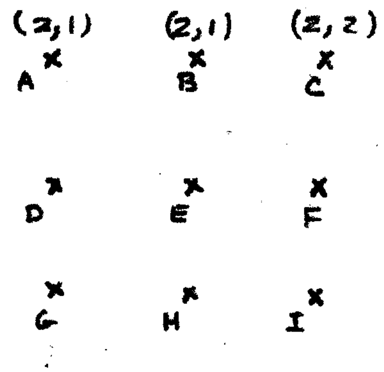


Figure 2

where $\underline{A},\underline{B},\underline{C},\underline{D},\underline{G}$ are respectively the computed correlation values at points $\bar{A}$, $\bar{B}$, C, D, G.

We can adjoint the constraints to the equation to form

$$H = J + \lambda_1(\underline{A}-g_A) + \lambda_2\left(P-\frac{\delta g}{\delta x}\Big|_A\right) + \lambda_3\left(Q-\frac{\delta g}{\delta y}\Big|_A\right)$$

taking partial derivatives and computed the coefficients in g which satisfy H.

If g(x'y) is polynomial in x, y, the set of equations one must solve become linear equations, and an analytical solution can be obtained.