LARS TECHNICAL REPORT 091577

A SIMPLIFIED DESIGN PROCEDURE FOR IMAGE RESTORATION AND ENHANCEMENT FILTERS

BY

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1. Introduction

The instantaneous field of view (IFOV) enhancement, developed [1/] at LARS primarily for improving resolution of Landsat imagery, is actually a kind of image restoration combined with interpolation to give larger pictures with sharper edges. This type of image restoration differs from other types [2/] in several aspects: first, it has precision control over the tradeoff between resolution gain and noise increase; second, its filter design is essentially data-independent, hence only one design is required for one imaging system; third, it allows shaping of the composite system function so that sidelobes can be reduced if desired. However, there are some drawbacks; first, the filter design technique, as used by McGillem and Riemer [20, 21], involves solving a system of differential equations by a digital computer and requires substantial amounts of CPU time (in the order of hours); moreover, the filter obtained has to be truncated to reduce size, thus it is only an approximation to the optimal. Second, in the continuous IFOV enhancement model, interpolation is only an add-on stage to the restoration filter, there is a strong chance that the restoration filter is not an optimal match to the interpolator. The present version of the IFOV

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1/: Developed by C.D. McGillem and T.E. Riemer, see references [20,21], the same problem was also studied by Smith [24].

2/: To name some other restoration methods: Wiener filtering [16,17], constrained minimum mean square estimate [5, 6], and some others, see references [7, 13, 18, 19].

enhancement filter is a trauncated, re-sampled version of the convolution of a cubic interpolation filter and a restoration filter using the discrete Fourier transform technique [22]. There may be some loss of the optimality which has been purchased so dearly in the design of the restoration filter. Third, the final operational IFOV enhancement filter has a size of 97 by 129 points which is such that the filter operation cannot take much advantage of either circular convolution or direct convolution. As a result, the enhancement requires a substantial amount of CPU time to process a full frame of Landsat data.

In this report a simplified solution method for the restoration filter (Stuller, [9, 10, 11]) is described together with still further simplifications which reduce the computer solution time requirement to the order of seconds. The restoration filter obtained by this method has certain optimal aspects which will be described later. Finally, an interpolation-restoration method that ensures optimality is described. The final optimal enhancement filter obtained is found to be significantly smaller and to have better performance than those previously obtained.

2. Design of the Restoration Filter

Figure 1 is a widely used model of image degradation and restoration which is based on a continuous shift-invariant blurring function $b(\underline{x})$ and a continuous restoration filter $p(\underline{x})$, where $\underline{x}$ is the two-dimensional coordinate system of the image. Let $f(\underline{x})$ and $n(\underline{x})$ be the original image and the noise introduced by the imaging system, then the blurred image can be expressed as:

$$g(\underline{x}) = \int_{-\infty}^{\infty} b(\underline{x}-\underline{u})f(\underline{u})d\underline{u} + n(\underline{x}) = b(\underline{x})*f(\underline{x})+n(\underline{x}) \qquad (1)$$

where * denotes the convolution of two functions and also the integration is understood to be over the entire two-dimensional space.

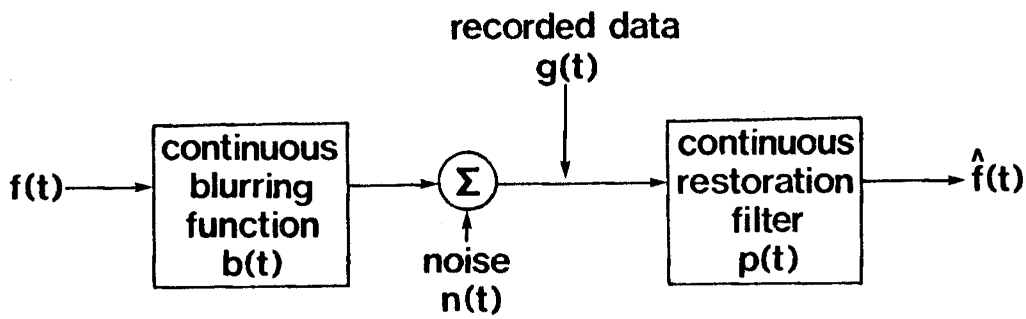


Fig. 1: A continuous-continuous model of image restoration

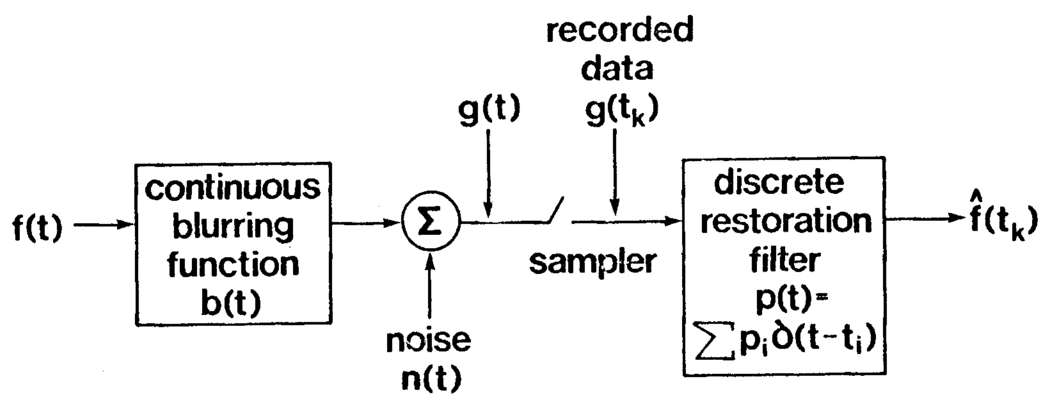


Fig. 2: A continuous-discrete model of image restoration

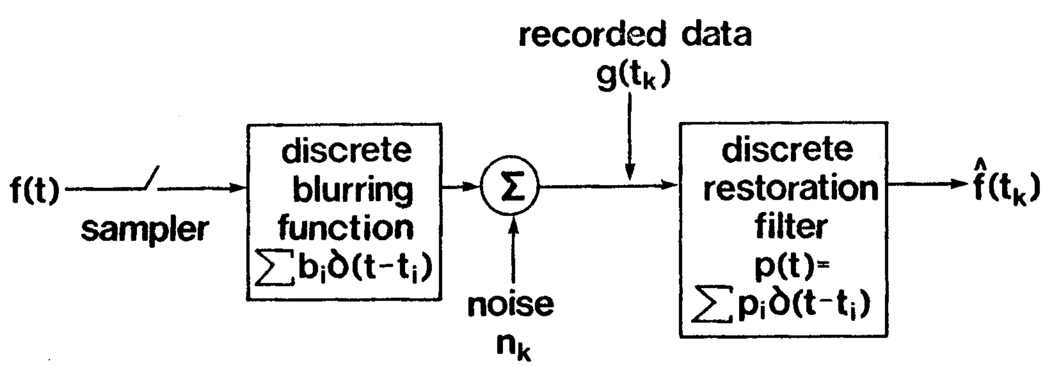


Fig. 3: A discrete-discrete model of image restoration

The restored image $f(\underline{x})$ is given by:

$$f(\underline{x}) = \int_{-\infty}^{\infty} p(\underline{x}-\underline{u})g(\underline{u})d\underline{u} = p(\underline{x})(b(\underline{x})*f(\underline{x}) + p(\underline{x})*n(\underline{x}) \quad (2)$$
$$= c(\underline{x})*f(\underline{x}) + n_o(\underline{x})$$

where $c(\underline{x}) = b(\underline{x})*p(\underline{x})$ is the composite point spread function of the overall degradation and restoration system and $n_o(\underline{x})=p(\underline{x})*n(\underline{x})$ is the output noise whose power is usually greater than the power of the original noise $n(\underline{x})$.

By resolution improvement, what is generally meant is that the width of the composite point spread function $c(\underline{x})$ is narrower than the width of the blurring function. However, there is no unique solution for the restoration filter $p(\underline{x})$ unless we express the 'width' of a function in a quantitative manner. There are several commonly used quantitative measures of duration, concentration or width of a function. One of these which is simple to use is the radius of gyration (Smith, [24]), R_c, of a function $c(\underline{x})$, defined as follows:

$$R_c^2 = \frac{\int_{-\infty}^{\infty} ||\underline{x}||^2 c^2(\underline{x})d\underline{x}}{\int_{-\infty}^{\infty} c^2(\underline{x})d\underline{x}} \quad (3)$$

where $||\underline{x}||$ is the distance function of point $\underline{x}$ from the origin.

In carrying our resolution improvement, it is generally necessary to restrict the amount of output noise so that the restored picture will not be dominated by noise. Mathematically, this is accomplished by restricting the output noise power to be less than some specified level σ^2:

$$E_{\underline{x}}\{n_o^2(\underline{x})\} \leq \sigma^2 \quad (4)$$

We can state the entire restoration problem in mathematical terms as follows:

Find a function $p(\underline{x})$ which minimizes the quantity

$$R^2 = \frac{\int_{-\infty}^{\infty} ||\underline{x}||^2 [b(\underline{x}) * p(\underline{x})]^2 \, d\underline{x}}{\int_{-\infty}^{\infty} [b(\underline{x}) * p(\underline{x})]^2 \, d\underline{x}} \qquad (5)$$

Subject to the output noise constraint:

$$E_{\underline{x}} \{[p(\underline{x}) * n(\underline{x})]^2\} \leq \sigma^2 \qquad (6)$$

The solution can be found through use of Lagrange multipliers and the theory of operators. Briefly the procedure to obtain a solution is as follows:

First form the functional

$$L = \int_{-\infty}^{\infty} ||\underline{x}||^2 [b(\underline{x}) * p(\underline{x})]^2 d\underline{x} - \lambda_1 \int_{-\infty}^{\infty} [b(\underline{x}) * p(\underline{x})]^2 \, d\underline{x}$$
$$+ \lambda_2 \, E_{\underline{x}} \{[p(\underline{x}) * n(\underline{x})]^2\} \qquad (7)$$

If the noise is stationary, the last term of (7) can be expressed in terms of the autocorrelation function of the noise $R_{nn}(\underline{\tau})$:

$$\lambda_2 \, E_{\underline{x}} \{[p(\underline{x}) * n(\underline{x})]^2\} = \lambda_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(\underline{u}) p(\underline{v}) \, R_{nn} \, (\underline{u} - \underline{v}) \, d\underline{u} d\underline{v} \qquad (8)$$

Now set the gradient of L with respect to $p(\underline{x})$ equal to zero and solve for $p(\underline{x})$. Using the technique of variational calculus described by Franks [23], the resulting expression is:

$$\lambda_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(\underline{u} - \underline{x}) b(\underline{u} - \underline{v}) p(\underline{v}) d\underline{u} d\underline{v}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ||\underline{u}||^2 b(\underline{u} - \underline{x}) b(\underline{u} - \underline{v}) p(\underline{v}) d\underline{u} d\underline{v} + \lambda_2 \int_{-\infty}^{\infty} R_{nn} (\underline{x} - \underline{u}) p(\underline{u}) d\underline{u} \qquad (9)$$

Since the bluring function $b(\underline{x})$ is known, the restoration filter $p(\underline{x})$ can be found by solving (9). The solution technique used by Smith and McGillem and Riemer involves taking the Fourier transform of both sides of (9), this leads to a second order differential equation with complex variables which must, in general, be solved by numerical methods which are to some degree approximations. An alternate and much simpler approach is to assume that the restoration filter $p(\underline{x})$ is discrete, then (9) becomes a matrix equation whose solution can be computed by readily available computer subroutines. This procedure is described in the next section.

3. Simplified Solution Method

The simplified solution method was described by Stuller [9, 10, 11]. The main idea is to reduce the admissible solution set to a smaller subspace. Instead of allowing the solution $p(\underline{x})$ to be any function, only discrete filters are permitted. It is easier to illustrate this in the one dimensional case. Consider a one dimensional discrete restoration filter $p(t) = \sum_{i=-n}^{n} p_i \delta(t-t_i)$, where the p_i's are the 2n+1 coefficients of the discrete filter and the t_i's are the time shifts to space the points of the filter. Now, the minimization problem which was stated in (5) and (6) becomes:

Determine the 2n+1 coefficients $p_{-n},\ldots,p_o,\ldots,p_n$

which minimize the quantity:

$$R^2 = \frac{\int_{-\infty}^{\infty} t^2 [b(t) * \sum_{i=-n}^{n} p_i \delta(t-t_i)]^2 dt}{\int_{-\infty}^{\infty} [b(t) * \sum_{i=-n}^{n} p_i \delta(t-t_i)]^2 dt} \qquad (10)$$

subject to the noise constraint

$$E\{[n(t) * \sum_{i=-n}^{n} p_i \delta(t-t_i)]^2\} \leq \sigma^2 \qquad (11)$$

With $c(t) = b(t)*[\sum_{i=-n}^{n} p_i \delta(t-t_i)] = \sum_{i=-n}^{n} p_i b(t-t_i)$, the numerator of the expression in (10) can be rewritten as:

$$\int_{-\infty}^{\infty} t^2 [\sum_{i=-n}^{n} p_i b(t-t_i)][\sum_{j=-n}^{n} p_j b(t-t_j)]dt$$

$$= \sum_{i=-n}^{n} \sum_{j=-n}^{n} p_i p_j \int_{-\infty}^{\infty} t^2 b(t-t_i) b(t-t_j) dt$$

$$= \underline{p}^T \underline{A} \underline{p} \qquad (12)$$

where the elements of $\underline{A}$ are given by

$$a_{ij} = \int_{-\infty}^{\infty} t^2 b(t-t_i)b(t-t_j)dt, \quad i,j = -n,\ldots,0,\ldots,n \tag{13}$$

Similarly for the denominator

$$\int_{-\infty}^{\infty}[\sum_{i=-n}^{n} p_i b(t-t_i)][\sum_{j=-n}^{n} p_j b(t-t_j)]dt = \underline{p}^T\underline{Bp} \tag{14}$$

where the elements of $\underline{B}$ are given by

$$b_{ij} = \int_{-\infty}^{\infty} b(t-t_i)\, b(t-t_j)dt, \quad i,j = -n,\ldots,0,\ldots,n \tag{15}$$

The left hand side of the noise constraint becomes:

$$E\{n_o^2(t)\} = E\{[\sum_{i=-n}^{n} p_i n(t-t_i)][\sum_{j=-n}^{n} p_j n(t-t_j)]\}$$

$$= \underline{p}^T\underline{Np} \tag{16}$$

where the elements of $\underline{N}$ are

$$n_{ij} = E\{n(t-t_i)n(t-t_j)\}, \quad i,j = -n,\ldots,0,\ldots n \tag{17}$$

The matrix $\underline{N}$ is the autocorrelation matrix of the noise.

The optimization problem can now be expressed as follows:

Minimize

$$R^2 = \underline{p}^T\underline{Ap}/\underline{p}^T\underline{Bp} \tag{18}$$

subject to the constraint

$$\underline{p}^T\underline{Np} \leq \sigma^2 \tag{19}$$

At this point, one may notice that the indices of these matrices and vectors run from -n to n. Also, from c(t) = b(t)*p(t), we have

$$\underline{p}^T\underline{Ap} = \int_{-\infty}^{\infty} t^2 c^2(t)dt \tag{20}$$

Since t^2 is always positive and $c^2(t)$ is always non-negative, it follows that $\underline{p}^T\underline{Ap} \geq o$ and $\underline{p}^T\underline{Ap} = o$ only when $c(t) \equiv o$, thus matrix $\underline{A}$ is positive-definite. A similar argument can be applied to $\underline{B}$, and $\underline{N}$, being an autocorrelation matrix, also has this property. Further, by examining a_{ij} and b_{ij}, it is seen that $a_{ij} = a_{ji} = a_{-i,-j} = a_{-j,-i}$ and similarly for b_{ij}. Thus, $\underline{A}$

and $\underline{B}$ are persymmetric matrices, i.e., they are matrices that are symmetric about each diagonal.

The denominator of (18) corresponds to the energy of the composite system function as can be seen from (10). Thus it is essentially a gain factor and does not alter the shape of the response function. By setting this quantity equal to unity the optimization problem can be restated as follows.

Minimize

$$R^2 = \underline{p}^T \underline{Ap} \tag{21}$$

subject to constraints

$$\underline{p}^T \underline{Bp} = 1 \tag{22}$$

$$p^T Np = \sigma^2 \tag{23}$$

A problem of this kind may be solved using Lagrange multipliers. However, it is necessary that the set

$$\{\underline{p}: \quad h_1 = \underline{p}^T \underline{Bp},\ h_2 = \underline{p}^T \underline{Bp}\} \tag{24}$$

satisfy so-called regularity conditions [25]. To do so, it is sufficient that the gradients $\underline{\nabla h_1}$ and $\underline{\nabla h_2}$ are independent. To check this, we impose a condition that σ^2 is not equal to any of the eigenvalues of $\underline{B}^{-1}\underline{N}$. Now suppose that $\underline{\nabla h_1} = 2\underline{Bp}$ and $\underline{\nabla h_2} = 2\underline{Np}$ are dependent. Then there exists a constant γ such that

$$2\ \underline{Bp} = 2\underline{Np}$$

$$\text{or } \gamma\underline{p} = \underline{B}^{-1}\underline{Np} \tag{25}$$

This means that γ is an eigenvalue of $\underline{B}^{-1}\underline{N}$ but

$$\sigma^2 = \underline{p}^T\underline{Np} = \underline{p}^T\gamma\underline{Bp} = \gamma$$

which says that σ^2 is an eigenvalue of $\underline{B}^{-1}\underline{N}$ and thus contradicts the original assumption, hence the gradients $\underline{\nabla h}$, and $\underline{\nabla h_2}$ are independent.

The condition that σ^2 is not an eigenvalue of $\underline{B}^{-1}\underline{N}$ does not represent a severe restriction. Since $\underline{B}$ and $\underline{N}$ are positive-definite matrices, the eigenvalues are positive and distinct. If it should happen that the desired noise level σ^2 is an eigenvalue of $\underline{B}^{-1}\underline{N}$ a small change in σ^2 will correct the situation.

The solution of the minimization problem is obtained by forming the augmented functional

$$F = \underline{p}^T\underline{Ap} - \lambda_1(\underline{p}^T\underline{Bp}-1) + \lambda_2(\underline{p}^T\underline{Np}-\sigma^2) \tag{26}$$

where λ_1 and λ_2 are Lagrange multipliers which are constants to be determined as part of the solution. The extreme value of (26) is now found by setting the gradient equal to zero and solving for $\underline{p}$.

$$\nabla F = 2\underline{Ap} - 2\lambda_1\ \underline{Bp} + 2\lambda_2\ \underline{Np} = \underline{0} \tag{27}$$

$$\lambda_1\underline{Bp} = (\underline{A} + \lambda_2\ \underline{N})\underline{p} \tag{28}$$

or

$$\lambda_1\underline{p} = (\underline{B}^{-1}\underline{A} + \lambda_2\underline{B}^{-1}\underline{N})\underline{p} \tag{29}$$

the constraints are

$$\underline{p}^T\underline{Bp} = 1 \tag{30}$$

$$\underline{p}^T\underline{Np} = \sigma^2 \tag{31}$$

To find the restoration filter (28), (30) and (31) must be solved simultaneously for $\underline{p}$, λ_1 and λ_2. This is done by assuming a value for λ_2 and solving (28) for its eigenvalues. The smallest eigenvalue corresponds to the correct solution for λ_1 as can be seen by multiplying (28) by $\underline{p}^T$ and substituting the values of the constraints.

$$\lambda_1\underline{p}^T\underline{Bp} = \underline{p}^T\underline{Ap} + \lambda_2\underline{p}^T\underline{Np}$$

i.e.,

$$R^2 = \lambda_1 - \lambda_2\sigma^2 \tag{32}$$

For any given λ_2 and σ^2 the smallest radius of gyration corresponds to the smallest λ_1. The solution for $\underline{p}$ obtained from the assumed λ_2 is substituted into (30) and (31) to see if they are satisfied. If not a new value of λ_2 is chosen and a new $\underline{p}$ computed. This is repeated until a solution is obtained that simultaneously satisfies all of the equations. The following algorithm is a summary of this:

Alg. 1: To compute a restoration filter by the simplified method

1. Make an initial guess of a value of the scalar λ_2, say $\lambda_2 = 1$
2. Compute eigenvectors $\underline{p}$ and eigenvalues λ_1 from $\lambda_1 \underline{Bp} = (\underline{A} + \lambda_2 \underline{N})\underline{p}$
3. Choose the smallest eigenvalue and its eigenvector $\underline{p}$
4. Compute the quantity $c^2 = \underline{p}^T \underline{Bp}$
5. Divide $\underline{p}$ by c (now $\underline{p}$ satisfies (30))
6. Compute the quantity $s^2 = \underline{p}^T \underline{Np}$. Check if $|s^2 - \sigma^2| <$ acceptable error, stop if yes; otherwise decrease λ_2 if $s^2 > \sigma^2$, or increase is $s^2 < \sigma^2$, go to step 2.

4. Technique to Reduce Computation

Since the matrices $\underline{A}$, $\underline{B}$ and $\underline{N}$ are persymmetric, it can be shown that the solution vector $\underline{p}$ is either symmetric or skewsymmetric [12]. The order of these matrices is 2n+1 which is odd and the eigenvector $\underline{p}$ corresponding to minimum radius of gyration is symmetric; that is, the coefficients $p_{-i}=p_i$ for all i = 1,2,...,n, and therefore there are only n+1 unknowns. Intuitively, it should be possible to solve for these unknowns from matrices of order n+1 only, and, if so, round-off error and computer time will be significantly reduced. This may be accomplished by deriving matrices $\underline{U}$, $\underline{V}$ and $\underline{S}$ of order n+1 corresponding to $\underline{A}$, $\underline{B}$ and $\underline{N}$ as described below. Starting with $p_{-i}=p_i$, let us rewrite the restoration filter p(t) as:

$$p(t) = \sum_{i=-n}^{n} p_i\delta(t-t_i) = p_o + \sum_{i=1}^{n} p_i[\delta(t-t_i) + \delta(t+t_i)] \tag{33}$$

Thus

$$b(t)*p(t) = p_o b(t) + \sum_{i=1}^{n} p_i[b(t-t_i) + b(t+t_i)] \tag{34}$$

$$= p_o\alpha_o(t) + p_1\alpha_1(t) + \ldots + p_n\alpha_n(t) \tag{35}$$

where

$$\alpha_i(t) = \begin{cases} b(t), & i = 0 \\ b(t-t_i) + b(t+t_i), & i = 1,2,\ldots,n \end{cases} \tag{36}$$

Now define a vector $\underline{r} = [p_o, p_1, \ldots, p_n]^T$ (37)

Then the numerator the expression of the radius of gyration in (10) may be rewritten as:

$$\int_{-\infty}^{\infty} t^2[b(t)* \sum_{i=-n}^{n} p_i\delta(t-t_i)]^2 dt$$

$$= \int_{-\infty}^{\infty} t^2[\sum_{i=0}^{n} p_i\alpha_i(t)][\sum_{j=0}^{n} p_j\alpha_j(t)]dt$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{n} p_i p_j \int_{-\infty}^{\infty} t^2\alpha_i(t)\alpha_j(t)dt$$

$$= \underline{r}^T\underline{Ur} \tag{38}$$

where the elements of $\underline{U}$ are given by

$$U_{ij} = \int_{-\infty}^{\infty} t^2\alpha_i(t)\alpha_j(t)dt, \quad i,j = 0,1,\ldots,n \tag{39}$$

$$= \begin{cases} a_{o,o}, & \text{if } i=j=0 \\ a_{o,-j}+a_{o,j}, & \text{if } i=0 \text{ and } j\neq 0 \\ a_{-i,o}+a_{i,o}, & \text{if } j=0 \text{ and } i\neq 0 \\ a_{i,-j}+a_{-i,j}+a_{i,-j}+a_{ij}, & \text{otherwise} \end{cases} \tag{40}$$

Similarly, the denominator of (10) may be rewritten as:

$$\int_{-\infty}^{\infty}[b(t)*\sum_{i=-n}^{n} p_i\delta(t-t_i)]^2dt = \underline{r}^T\underline{Vr} \tag{41}$$

where the elements of $\underline{V}$ are given by

$$v_{ij} = \int_{-\infty}^{\infty}\alpha_i(t)\alpha_j(t)dt,\ i,j = 0,2,\ldots,n \tag{42}$$

$$= \begin{cases} b_{o,o}, & \text{if } i=j=o \\ b_{o,-j}+b_{o,j}, & \text{if } i=0 \text{ and } j\neq 0 \\ b_{-i,o}+b_{i,o}, & \text{if } j=0 \text{ and } i\neq 0 \\ b_{i,-j}+b_{-i,j}+b_{i,-j}+b_{ij}, & \text{otherwise} \end{cases} \tag{43}$$

The left hand side of the noise constraint may be rewritten in an analogous fashion as:

$$E\{n_o(t)\} = E\{[\sum_{i=o}^{n} p_i\eta_i(t)][\sum_{j=o}^{n} p_j\eta_j(t)]\}$$

$$= \underline{r}^T\underline{Sr} \tag{44}$$

where $\eta_i(t) = \begin{cases} n(t), & i=0 \\ n(t-t_i) + n(t+t_i), & i = 1,2,\ldots,n \end{cases}$ (45)

and the elements of $\underline{S}$ are given by

$$s_{ij} = E\{\eta_i(t)\eta_j(t)\} \tag{46}$$

$$= \begin{cases} n_{o,o}, & \text{if } i=j=0 \\ n_{o,-j}+n_{o,j}, & \text{if } i=0 \text{ and } j\neq 0 \\ n_{-i,o}+n_{i,o}, & \text{if } j=0 \text{ and } i\neq 0 \\ n_{i,-j}+n_{-i,j}+n_{i,-j}+n_{ij}, & \text{otherwise} \end{cases} \tag{47}$$

A straight-forward application of the Lagrange multiplier method will show that the optimum vector $\underline{r}$ satisfies:

$$\gamma_1 \underline{V}\underline{r} = (\underline{U} + \gamma_2 \underline{S})\underline{r} \tag{48}$$

$$\underline{r}^T \underline{V}\underline{r} = 1 \tag{49}$$

$$\underline{r}^T \underline{S}\underline{r} = \sigma^2 \tag{50}$$

The matrices $\underline{U}$, $\underline{V}$ and $\underline{S}$ are of order (n+1) only, so computation is significantly reduced. It should be noted that the last three equations are analogous to equations (28), (30), (31), thus the solution $\underline{r}$ may be computed by the same algorithm as given by Alg. 1. By expanding the n+1 elements of vector $\underline{r}$ to 2n+1 elements, the restoration filter $\underline{p}$ is then obtained.

5. Discussion of the Simplified Solution Method

If the restoration filter has a short duration, then filtering of a large picture can be efficiently performed by direct convolution[1/]. One of the goals in the original McGillem and Riemer filter design was to concentrate the restoration filter near the origin so that a short truncated version of it could be used for filtering to reduce edge effects as well as reduce computation time. This was achieved by introducing an extra constraint on the width of the filter. The price of doing this was added difficulty of solving the system of differential equations and only reduced but not eliminated the filter amplitude away from the origin. The simplified method described above permits arbitrary selection of the filter's length and the solution is still guaranteed to be optimal for that length of the restoration filter. This optimality property provides an optimal but simple means of achieving the desired goal. Furthermore, the size of the

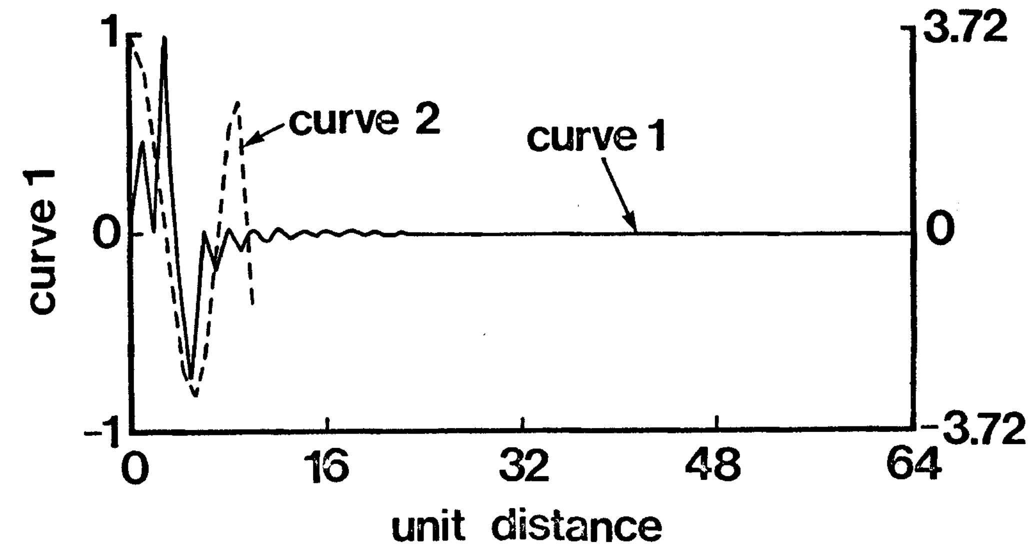
1/: Filtering of large pictures, e.g., 2048 by 2048 pixels, will be easier by direct convolution if the filter is small. Circular convolution via the fast Fourier transform is difficult because the 2-dimensional picture has too much data to reside in the memory of an ordinary computer.

the matrices depends on the length of the filter, so no CPU time is wasted to compute some points of the filter which are eventually to be discarded. This is part of the reason that this method takes much less CPU time to carry out the design.

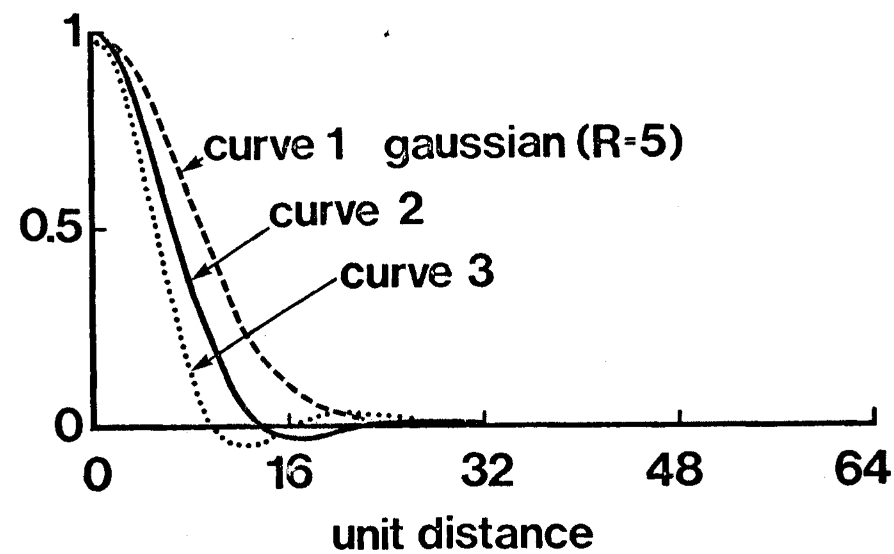
The discrete restoration filter is a more realistic model for image degradation and restoration for certain applications than is the continuous representation. Figure 2 shows an imaging system similar to that of Landsat. In this system the optics degrades the original picture it takes, and a quantizer introduces noise while it converts the analog or continuous picture to a digital picture. The resulting output image is sampled and quantized and has additive noise. The restoration filter operates on the discrete image. For such a situation the discrete restoration filter is more suitable than the continuous filter.

Another advantage of the simplified procedure is that it can be readily extended to the completely discrete image degration and restoration model, as shown in Figure 3. Such a model has an advantage of not requiring knowledge of the functional form of the blurring function, only the tabulated values are required. This is especially suitable for imaging systems whose blurring function is not exactly known, but can be determined empirically by other image processing techniques.

The simplified method gives solutions for restoration filters very similar to those obtained by solving the differential equations. A one-dimensional continuous guassian blurring function of radius of gyration 5 (standard deviation is 7.071) is chosen to illustrate this similarity. Curve 1 of Figure 4 (a) is the restoration filter obtained by solving the differential



(a) Restoration filters designed for a guassian blurring function with a radius of gyration of 5. Curve 1 is the continuous restoration filter obtained by solving the differential equation, curve 2 is the discrete restoration filter obtained by the simplified solution method.



(b) Composite system functions for the restoration filters in (a). Curve 1 is the guassian blurring function, curve 2 corresponds to that obtained by solving the differential equations, curve 3 to that computed by simplified solution method.

Fig. 4: Comparison of restoration filters and their corresponding composite system functions obtained by various method of solution.

equation (McGillem and Riemer, Figure 4.79, p.111, reference [21]), this filter has a full length of 129 points, but if a 1% truncation threshold level is used, its length can be reduced to 33 points. The composite system function corresponding to this filter is curve 2 of Figure 4(b). The stated radius of gyration of this filter is 3.55 giving a ratio of 3.55/5=70.1% that of the original blurring function. Curve 2 of Figure 4(a) is a similar (in the sense that it has about the same noise increase) restoration filter obtained by the simplified method. The length of this filter is chosen to be 21 points so that it is shorter than the other filter. The composite system function corresponding to this filter is curve 3 of Figure 4(b). This filter has a radius of gyration of 3.25 giving a ratio of 3.25/5=65%. The shape of the simplified discrete filter is similar to that of curve 2 and both of them have noticeable sidelobes.

Table 1 is a tabulation of CPU time required to compute some discrete restoration filters using the simplified method. The time requirement depends (quadratically) on the length of the filter. It should be noted that the CPU time requirements is reduced 3 times if the filter is computed from the smaller matrices $\underline{U}$, $\underline{V}$ and $\underline{S}$. The saving becomes more noticeable for large filters. However, as mentioned before, the CPU time requirement is generally only in the order of seconds.

6. An Optimal Interpolation-Restoration Processor for IFOV Enhancement

Figure 5 is the schematic diagram of the IFOV enhancement for a one-dimensional signal. The reason for treating the two-dimensional pictorial data as one-dimensional signals is that the single dimension case is much easier to deal with both in the filter design and in the filtering operation. The front part of this model is a typical digital image-gathering

Table 1: Time requirement for computing various filters using the simplified method on the CDC 6500 computer. It is estimated that it would be 2-4 times slower on the IBM 360/67 computer.

Length of Restoration Filter	(From Matrices A, B and N) Continuous-Discrete Model	Discrete-Discrete Model	(From Matrices U, V and S) Continuous-Discrete Model	Discrete-Discrete Model
3	.8(seconds)	9.6	.04(seconds)	.5
5	1.1	10.5	.11	1.0
7	1.5	14.6	.20	1.9
9	2.1	19.3	.36	3.4
11	2.9	26.1	.57	5.2
13	3.7	36.3	.80	7.8
15	4.8	44.8	1.2	11.0
17	6.0	55.8	1.5	14.7
19	7.4	70.8	2.1	19.8
21	9.2	90.3	2.6	26.4
23	11.1	104.8	3.4	32.2
25	13.4	126.2 (Estimated)	4.3	40.5

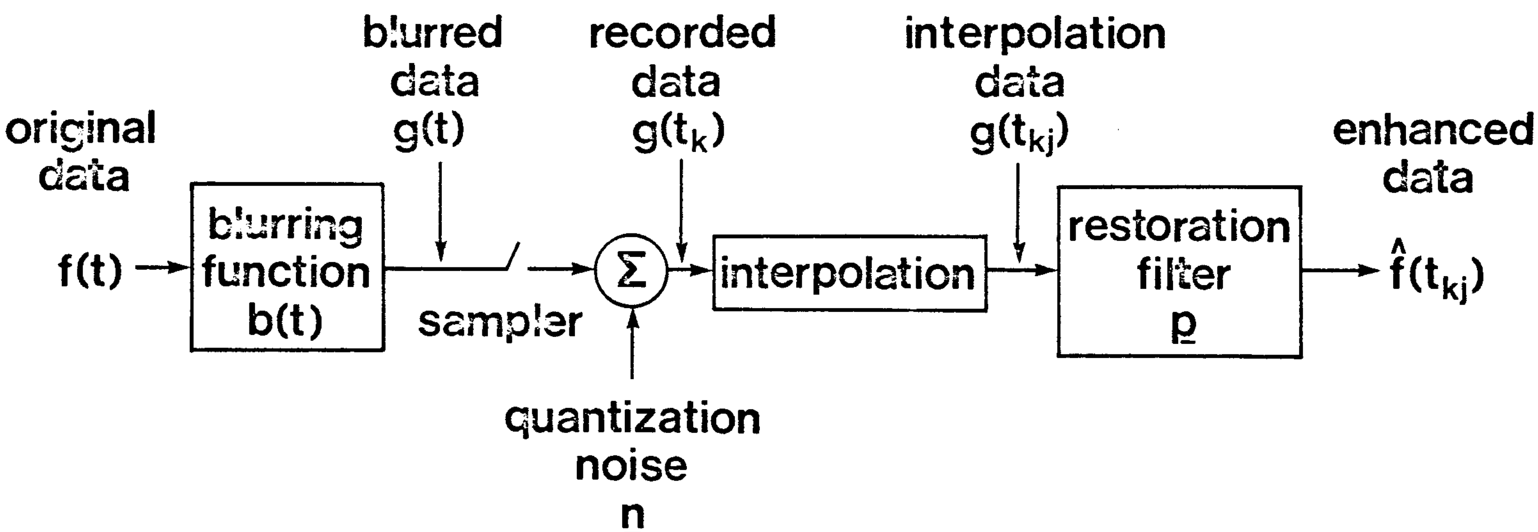


Fig. 5: A model of the interpolation and restoration process (for each dimension of the spatial domain) for IFOV enhancement.

system: an imaging degrading or blurring function, followed by a sampler to convert the analog image signal to digital data and thus add quantization noise to the signal. The middle part is an interpolation operation that computes intermediate data points to provide an enlarging effect. An interpolation scheme such as cubic interpolation can be regarded as convolution of zero-augmented data and an interpolating pulse [26]. Figure 6(a) is an illustration of cubic interpolation with a magnification factor of 4. The interpolating pulse is given by [26]:

$$\underline{h} = (0,-5,-8,-7,0,35,72,105,128,105,72,35,0,-7,-1,=5)/128 \qquad (51)$$

Figure 6(b) is a plot of $\underline{h}$ with equal sampling intervals.

The heart of the enhancement processor is a restoration filter which will depend up on both the blurring function and the interpolating pulse. In order to have an optimal restoration effect, the restoration filter must be designed taking into consideration the kind of interpolation that is to be employed; however, the model as shown in Figure 5 does not fit into one of the standard image degradation-restoration models (Figures 1,2,3) unless the blurring function is convolved with the interpolating pulse to produce an equivalent blurring function. Since sampling and zero-augmentation are involved, the discrete model of Figure 3 is preferrable. Let the vector $\underline{b}$ denote the sampled version of the blurring function b(t), i.e., each element of $\underline{b}$ is $b_i=b(t_i)$ where t_i is the sampling instant (do not confuse b_i with b_{ij} in (15) which is an element of the square matrix $\underline{B}$), and let the chosen magnification factor be M. The zero-augmented sequence is given by

$$\underline{b}_A = (\ldots,\ b_{-1},0,\ldots,0,b_o,0,\ldots,0,b_1,0,\ldots,0,\ldots) \qquad (52)$$

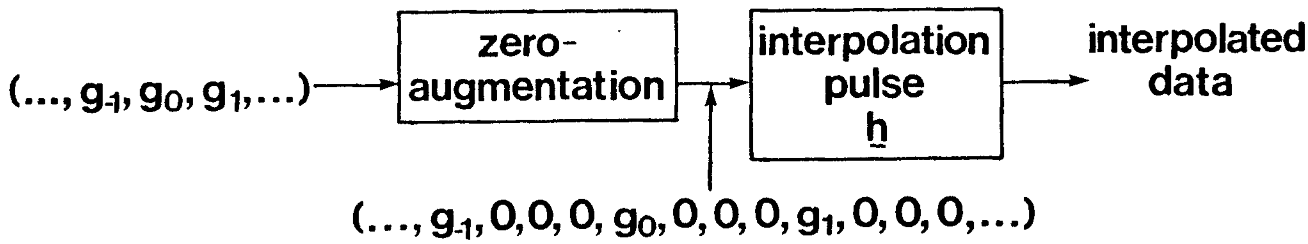


Fig. 6(a): An interpolation may be regarded as convolution of the zero-augmented data and an interpolating pulse which characterizes the type of interpolation.

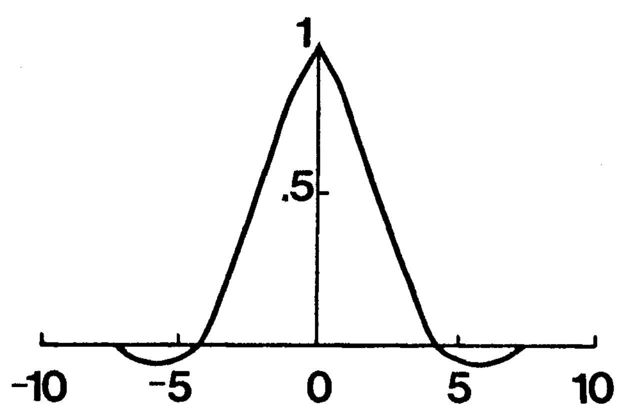


Fig. 6(b): An example of the cubic interpolating pulses. The shown pulse has a magnification factor of 4.

with M-1 zeros between each element b_i of vector $\underline{b}$. Now the equivalent blurring function $\underline{b}_e$ can be formed as:

$$\underline{b}_e = \underline{b}_A * \underline{h} \tag{53}$$

Such a convolution will also be denoted by

$$b_{ek} = b_{Ak} * h_k \tag{54}$$

with k regarded as a discrete variable.

The equivalent noise $\underline{n}_e$ can be similarly formed by zero-augmentating the noise sequence $\underline{n}$ and then convolving with the interpolating pulse:

$$\underline{n}_e = \underline{n}_A * \underline{h} \tag{55}$$

With these equivalent functions, the enhancement model fits into the discrete-discrete degradation-restoration model as shown in Figure 7. What is now required is to compute the square matrices $\underline{A}$,$\underline{B}$, and $\underline{N}$ (we call these $\underline{A}_e$, $\underline{B}_e$ and $\underline{N}_e$ hereafter). These matrices are defined in terms of the continuous functions in (13), (15) and (17), but with a slightly modified derivation. The discrete counterparts can be found as follows:

$$a_{eij} = \sum_{k=-\infty}^{\infty} (k\Delta t)^2 \; b_{e(k-i)} b_{e(k-j)} \Delta t \tag{56}$$

$$b_{eij} = \sum_{k=-\infty}^{\infty} b_{e(k-i)} b_{e(k-j)} \Delta t \tag{57}$$

$$n_{eij} = E_k\{n_{e(k-i)} n_{e(k-j)}\} \tag{58}$$

where b_{ei} and n_{ei} denote each element of the vectors $\underline{b}_e$ and $\underline{n}_e$, and a_{eij}, b_{eij}, and n_{eij} are the elements of matrices $\underline{A}_e$, $\underline{B}_e$, and $\underline{N}_e$.

The last equation can be converted into a more useful form as follows. Since $n_{ek}=n_{Ak}*h_k$, and $\underline{h}$ has 2L+1 elements, it is possible to rewrite this as

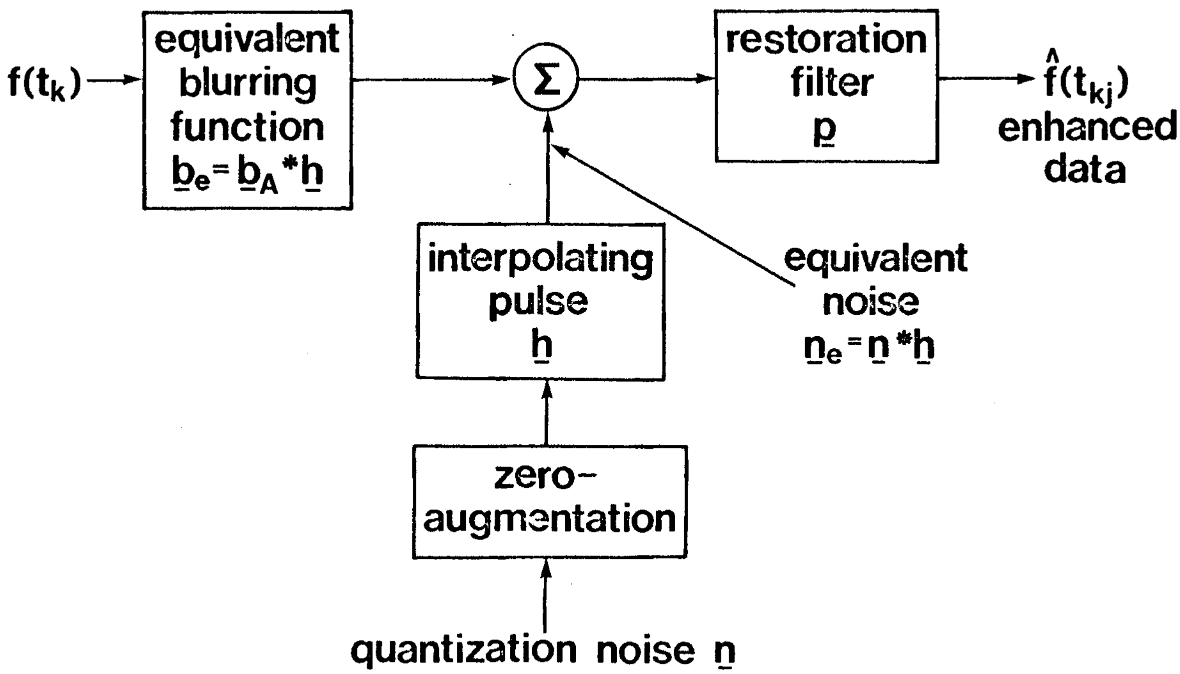


Fig. 7: An equivalent model of Fig. 5. This model is suitable for designing the restoration filter of the optimal IFOV enhancement processor.

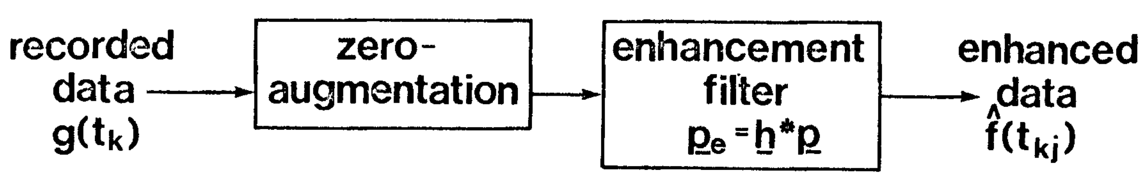


Fig. 8: Schematic diagram of the optimal IFOV enhancement processing.

$$n_{ek} = \sum_{l_1=-L}^{L} n_{A(k-l_1)} h_{l_1} \tag{59}$$

Substituting this into eq. (55), gives

$$n_{eij} = E_k\{(\sum_{l_1=-L}^{L} n_{A(k-i-l_1)} h_{l_1})(\sum_{l_2=-L}^{L} n_{A(k-j-l_2)} h_{l_2})\}$$

$$= \sum_{l_1=-L}^{L} \sum_{l_2=-L}^{L} E_k\{n_{A(k-i-l_1)} n_{A(k-j-l_2)}\} h_{l_1} h_{l_2} \tag{60}$$

The left hand side can be easily evaluated if it can be determined what the value the expectation will be at shifts $m_1=i+l_1$ and $m_2=j+l_2$. If the magnification factor M is greater than one, at least some zeros are augmented and some shifts will yield an expectation value of zero except when m_1-m_2 is a multiple of M. The expectation can therefore be written as:

$$E_k\{n_{A(k-m_1)}\ n_{A(k-m_2)}\}$$

$$= \begin{cases} E_k\{n_{k-m_1/M} n_{k-m_2/M}\} & \text{if } (m_2-m_1)/M=0,\pm 1,\pm 2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{61}$$

For many kinds of noise the stationarity assumption is valid, i.e.,

$$E_k\{n_{k-m_1/M} n_{k-m_2/M}\} = E_k\{n_k n_{k-(m_2-m_1)/M}\} \tag{62}$$

For this case, (61) becomes

$$E_k\{n_{A(k-i-l_1)}\ n_{A(k-j-l_1)}\}$$

$$= \begin{cases} E_k\ \{n_k n_{k-(m_2-m_1)/M}\} & \text{if } (m_2-m_1)/M = 0,\ \pm 1,\ \pm 2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{63}$$

If the original noise n_k is white, then $E_k\{n_k n_{k-(m_2-m_1)/M}\}$ is zero except when $m_1=m_2$, i.e., $l_1=l_2+i-j$, and eqt (58) can be simplified to

$$n_{eij} = (N_o/2)\left(\sum_{l_2=-L}^{L} h_{l_2} h_{l_2+i-j}\right) \tag{64}$$

where $N_o/2$ is the noise level.

Fortunately the set up of these matrices can be programmed as an algorithm for any given blurring function in tabulated form and any given interpolative pulse with any given magnification factor. The restoration filter $\underline{p}$ can be computed easily as outlined in Sections 4 and 5.

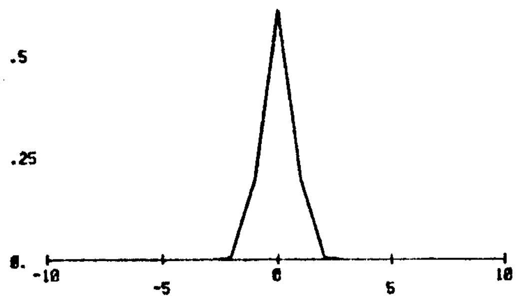
Once the restoration filter $\underline{p}$ is obtained, it can be combined (by convolution) with the interpolating pulse to yield the enhancement filter $\underline{p}_e$:

$$\underline{p}_e = \underline{h}*\underline{p} \tag{65}$$

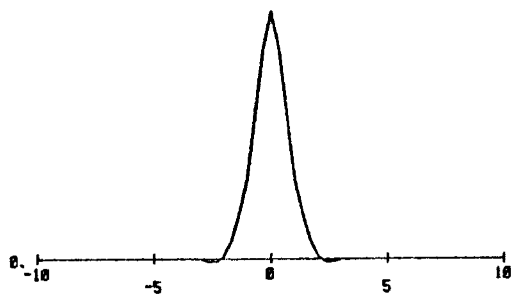
There is no need to interpolate the data first and then restore afterwards. Such a filter is optimal with respect to the minimum radius of gyration criterion and the selected interpolation scheme. The size of this filter can be controlled by carefully considering size reduction as a tradeoff with increased radius of gyration. Figure 8 is the schematic diagram of the enhancement processor and Figure 9 is an illustration of the design of an enhancement filter.

7. Preliminary Enhancement Experiements with Landsat Images

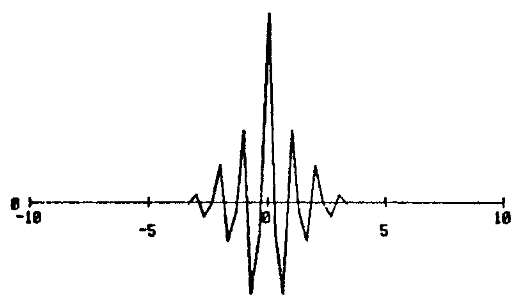
In these experiments, the ratio of radius of gyration of the composite system function to the radius of gyration of the blurring function is used as a performance indicator. The smaller such a ratio, the better is the resolution improvement. In the absence of the noise constraint, most one-dimensional filters have a ratio of about 0.5, however if output noise has to be restricted, most ratios are roughly 0.6–0.75. Even with a ratio of 0.707 for each dimension, the two-dimensional resolution element is reduced to 0.5 of its original size, and the improvement will be very



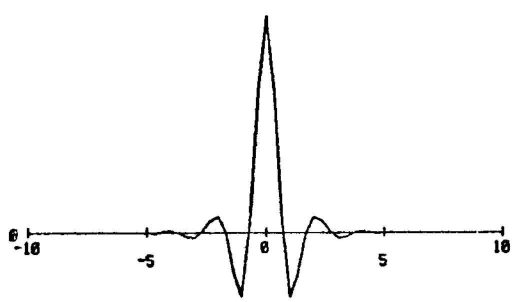
(a) A discrete version of the continuous gaussian blurring function having a radius of gyration of 0.4166.



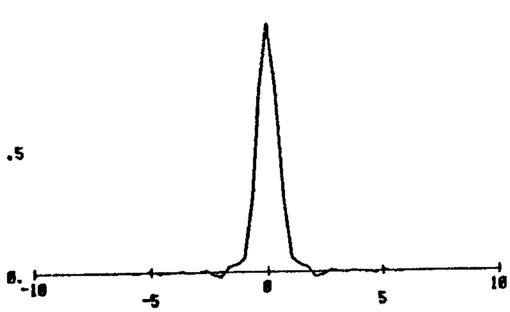
(b) 3-X cubic-interpolated version of the discrete guassian blurring function, formed by $\underline{b}_e = \underline{b}_A * \underline{h}$. It has a radius of gyration of 0.4918.



(c) A 21-point restoration filter $\underline{p}$ designed optimally for $\underline{b}_e$ in (b), noise increase is restricted to 22 dB.



(d) The enhancement filter $\underline{p}_e$, formed by $\underline{p}_e = \underline{p} * \underline{h}$.



(e) The composite system function $\underline{c}$ of the entire blurring and enhancement system, formed by $\underline{c} = \underline{b}_e * \underline{p}$. It has a radius of gyration of 0.312, a performance ratio of 63%.

Fig. 9: An example of the design of an optimal IFOV enhancement processor.

noticeable. The continuous IFOV enhancement filter was stated to have a ratio of 0.5 (two dimension). Its performance on a selected area near O'Hare Airport of Chicago is shown in Figure 10(c). It is seen that there is a clearly noticeable resolution improvement over the corresponding cubic interpolated picture (Figure 10(b)).

Using the optimal method described above, two enhancement filters were designed. The first filter had a two-dimensional performance ratio of 0.65 with about a 17 dB noise increase. This filter had a size of 31 by 35 points, only about 8.7% of the size of the continuous IFOV enhancement filter (97 by 129). Its performance on the same area is shown in Figure 10(d). By looking at the highways around the airport, it appears that its quality is slightly better than Figure 10(c).

The second filter that was tested had a two-dimensional performance ratio of 0.45, a noise increase roughly 22 dB and had a size of 31 by 35 points. As shown in Figure 10(e), the resolution improvement is more evident and there are as well as some undesirable effects of noise.

It may be concluded from these tests that the new enhancement filter possesses equal or better performance than the continuous model but has much smaller size. This will reduce filtering cost substantially.

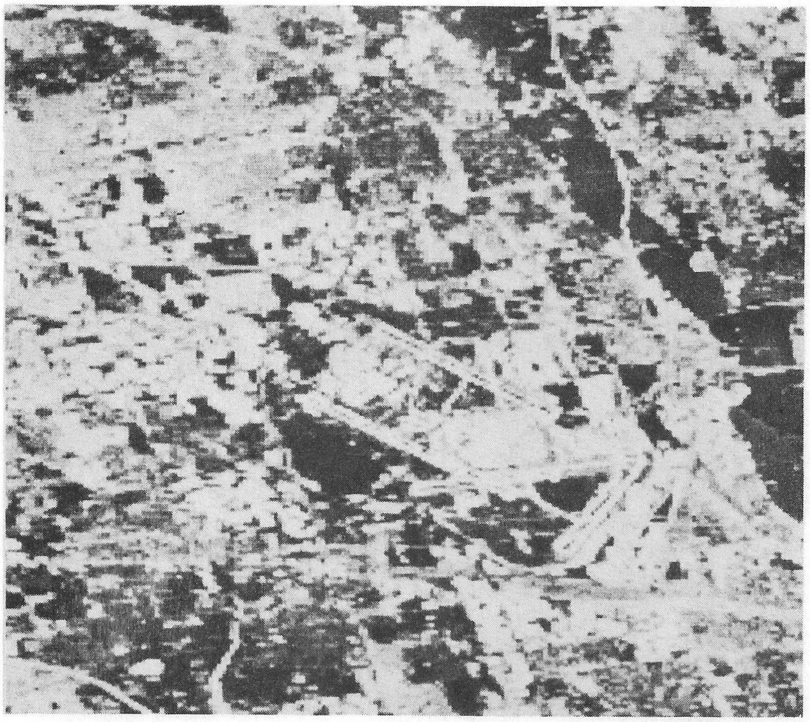


Fig. 10(a): The O'hare Airport, Chicago. Landsat imagery, LARS run no. 73033500, line 1001-1200, col 901-1100, channel 2.

Fig. 10(b): 3x4 cubic interpolation of Fig. 10(a).

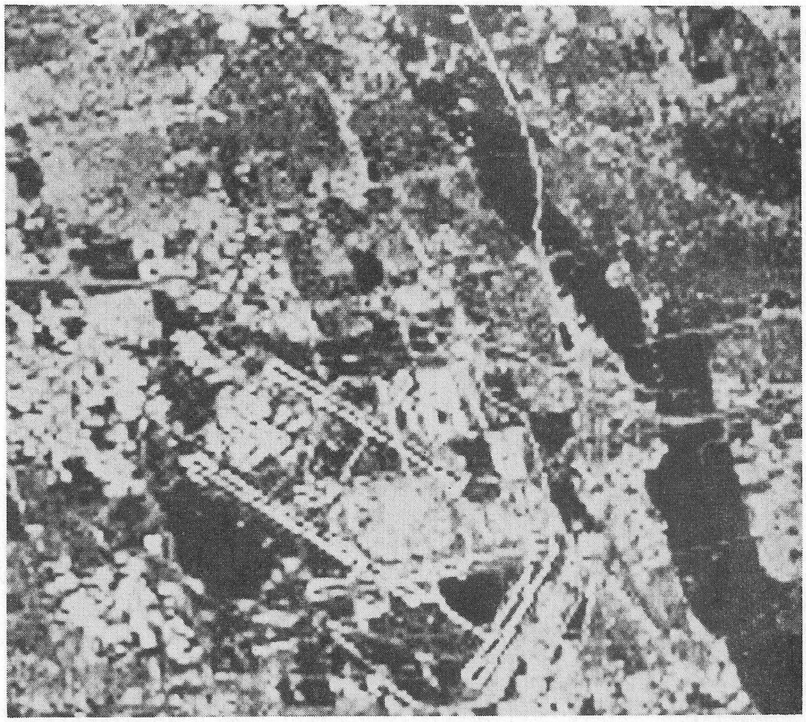


Fig. 10(c): The present continuous IFOV enhancement of Fig. 10(a).

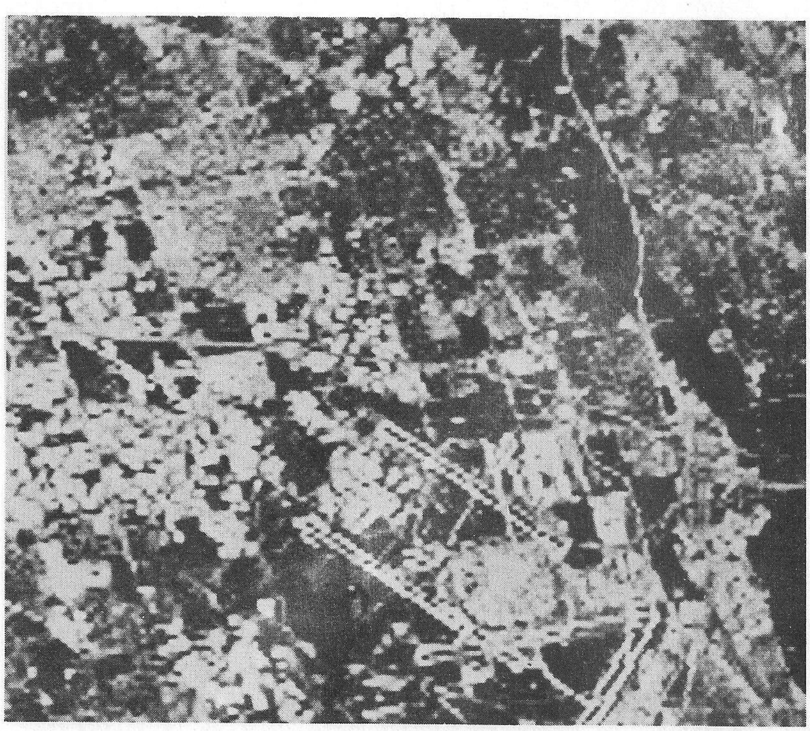


Fig. 10(d): An optimal IFOV enhancement of Fig. 10(a) using a restoration filter with a performance ratio of 0.65 and noise increase of 17 dB.

Fig. 10(e): An optimal IFOV enhancement of Fig. 10(a) using a restoration filter with a performance ratio of 0.45 and noise increase of 22 dB.

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