# WEIGHTING FUNCTION TECHNIQUES FOR STORAGE AND ANALYSIS OF MASS REMOTE SENSING DATA*

James R. Jancaitis and John L. Junkins

Division of Academic Computing and Department of Engineering
Science and Systems, University of Virginia
Charlottesville, Virginia

## I. ABSTRACT

Recent publications have documented the development and successful application of weighting function techniques for analytically modeling discretely measurable two-dimensional irregular surfaces. The weighting function techniques produce an analytic functional model valid over the entire input data set. The globally valid function is comprised of an arbitrarily large family of locally valid functions which join together with m-th order continuity assured. The locally valid functions are typically low order polynomials, so that analysis of the functional model is efficient and inexpensive.

The theory of the weighting function techniques has been extended to N-dimensional functions, allowing mathematical modeling of discretely measurable functions of an arbitrary number of independent parameters, with m-th order continuity assured. Analytic formula for the weighting function coefficients for N-dimensional, m-th order continuity have been developed.

This technique has been employed to store and analyze mass remotely sensed topographic data for contour maps and three-dimensional graphical displays.

## II. INTRODUCTION

The problem considered in this paper is the mathematical modeling of discretely measurable N-dimensional geodetic functions, e.g. given the observations;

$$\phi_i \text{ at } (x_{1i}, x_{2i}, x_{3i}, \ldots, x_{Ni}) \text{ for } i = 1,2,3,\ldots,P; \tag{1}$$

form an accurate m-th order continuous functional model

$$\phi = f(x_1, x_2, x_3, \ldots, x_N) \tag{2}$$

which reliably predicts the measured function at any desired point in the N-space.

*This work was supported by the U. S. Army Engineer Topographic Laboratories, Fort Belvoir, Virginia, under contract DAAK02-72-C-0256. Portions of this paper were included in "Modeling N-dimensional Surfaces using a Weighting Function Approach," a paper presented to the Fifty-Fourth Annual Meeting of the American Geophysical Union, Geodesy section, in Washington, D. C., April 16-20, 1973.

The primary motivation for formulating an analytic model of a discretely measured function is the flexibility and economy of this representation of the data. Accessing and analyzing an analytic model for information is significantly more efficient than accessing the usually dense, heavily redundant geodetic data set. New and different analyses of geodetic data often proceed more simply and directly from mathematical models. Also, storage of the data in functional form typically results in a significant decrease in physical storage requirements.

The desirable consequences proceeding from representation of geodetic surfaces with mathematical models has resulted in this problem receiving considerable attention in the literature. For observations of a function of one variable, e.g.

$$\phi_i \text{ at } (x_{1i}) \text{ for } i = 1, 2, 3, \ldots, p \tag{3}$$

the list of interpolation and approximation techniques is almost endless, the more classical procedures are covered in Davis (1963). The development of spline interpolation, Greville (1969), has allowed interpolation subject to a "best approximation" criterion. Unfortunately, the standard techniques result in unweildy functions or have numerical difficulties for the typically large multi-dimension geodetic data sets. For functions of one variable, Junkins, et al. (1972a) and Jancaitis and Junkins (1973b) developed sequential, non-singular, uniquely defined interpolation and approximation techniques for arbitrarily large data sets, which also guarantee arbitrary order continuity. Considering discretely measurable functions of two independent variables, e.g.

$$\phi_i \text{ at } (x_{1i}, x_{2i}) \text{ for } i = 1, 2, 3, \ldots, p \tag{4}$$

the classical and spline techniques generalize--but the difficulties encountered in one dimension are compounded in two. For practical models of geoidal scale surfaces in spherical coordinates, Lee and Kaula's (1967) spherical harmonic series and Lundquist and Giacaglia's (1971) sampling function series have been found to be useful and accurate mathematical representations. For the fine structure modeling of irregular surfaces, Jancaitis and Junkins (1973a,b) and Junkins et al., (1972b) presented sequential surface averaging methods based on locally valid polynomials. These techniques resulted in m-th order continuous analytic models for arbitrarily large data sets. Probably the most widely used mathematical model of a geodetic function of three variables, e.g.

$$\Phi = f(x_1, x_2, x_3) = f(\phi, \lambda, \gamma) \tag{5}$$

is the spherical harmonic expansion representation of the gravitational potential of the earth. For the macro-scale modeling of gravity, the spherical harmonic representation provides an accurate, efficient, continuous and globally valid model. A promising technique for modeling the fine structure variations of the earth's gravity field has been developed and demonstrated by Morrison (1972); it is based on local expansions of a geodial density layer.

Considering the applicability, advantages and shortcomings of these techniques, formulation of requirements for a generally applicable functional modeling technique is possible. A generalized mathematical modeling technique is desired which embodies the following characteristics; the mathematical model should:

1) approximate local data in a weighted least square sense, consistent with absolute and relative measurement precision,

2) permit efficient and accurate calculation of its determining parameters,

3) have any desired order continuity,

4) make use of any apriori knowledge of the nature of the geodetic function being modeled,

5) be so formulated that access and utilization for geodetic analyses be efficient, accurate and direct,

6) be scale independent, to allow modeling of either macro- or micro-scale variations,

7) be formulated in "generalized coordinates" so that the formulation will directly apply to representation of surfaces in any convenient N-space,

8) be easily utilized for arbitrarily large geodetic data sets,

9) permit efficient updating of the defining parameters based on new observation sets,

10) allow significant data compaction for more economical storage.

With these requirements in mind the modeling technique of this paper has been developed and demonstrated. The weighting function techniques detailed herein allow the mathematical modeling of a general N-dimensional discretely measureable geodetic function with m-th order continuity, for arbitrarily large data sets. This technique embodies the ten desired characteristics stated above. The technique is based upon a new family of polynomial weighting functions; analytic formulae for the generalized n-variable weighting functions' coefficients have been derived and are given herein.

For background and as a suitable base for the subsequent results of this paper, the one-dimensional weighting function interpolation technique will be discussed first.

## III. THE ONE-DIMENSIONAL WEIGHTING FUNCTION INTERPOLATION TECHNIQUE (WIT)

The one-dimensional problem can be stated as:

Given p observations of a discretely measurable function, $\Phi$, of one independent variable, x;

$$\Phi_i \text{ at } x_i \text{ for } i = 1, 2, 3, \ldots, p. \tag{6}$$

Find an accurate, m-th order continuous functional model, f, of the variable, $\Phi$,

$$\Phi = f(x) \tag{7}$$

which reliably predicts the function between its measured values.

The weighting function interpolation and approximation technique (WIT), which solves this problem, proceeds as follows:

1) choose some form for a sequential set of local approximations, based on experience or apriori knowledge of the nature of the geodetic function e.g. choose

$$F_j(x, p_1^j, p_2^j, p_3^j, \ldots, p_n^j), \tag{8}$$

to approximate $\Phi$ in a local neighborhood of a "centroid of validity" $\bar{x}_j$. With $F_j$ centered at the point $\bar{x}_j$ and the points $\bar{x}_j$ not necessarily equal to any of the $x_i$. The $p_k^j$'s are the constants of the local functional approximation, which are determined for each local neighborhood individually. (Taylor series have been found to work well for modeling topography.)

2) choose a set of sequential points $\bar{x}_j$, arbitrary in number and location, to serve as centroids of validity for the local functional approximations, $F_j$. The distribution and number of these points will be determined by the location and

densities of the original data, their known measurement accuracies, and the complexity of the local approximation's functional form.

3) employing standard numerical techniques, calculate the defining parameters of the local approximation functions. The original data can be fit exactly, in least square fashion, or using minimum criterion, etc., to determine the initial local approximations (8) associated with each $\bar{x}_j$. The arbitrariness of the form and determination of the local approximations affords a new level of flexibility in approximation.

4) use "appropriate" weighting functions to merge (average) the local approximations into a single m-th order continuous functional model valid over the entire original data set. This is accomplished by defining the globally valid functional model as a series of piecewise functions; each valid between two neighboring centroids of validity. To maintain the m-th order continuous nature of the global representation, it is clear that adjacent piecewise functions must join with m-th order continuity satisfied at their junction.

Of course the key to the success and applicability of WIT depends on the form of the final interpolating function (7) and therefore upon the properties of the weighting functions utilized to merge the local functional approximations. Of the many possible forms, the authors have chosen to employ

$$\phi_j(x) = F_j(x)J_j(x) + F_{j+1}(x)J_{j+1}(x) \tag{9}$$

to represent the geodetic function, $\phi$, in the intervals

$$\bar{x}_j \leq x \leq \bar{x}_{j+1}. \tag{10}$$

The $J_j(x)$ are the yet to be defined weighting functions. Other forms (and associated weighting functions) are under investigation and will be discussed in subsequent papers. This approach is unique because <u>functions</u> (as opposed to discrete values) are being interpolated.

The requirement that the piecewise function (9) form an m-th order continuous, globally valid model, and the additional requirements of reasonable interpolation at the endpoints and in the interior of the intervals lead to the boundary value problems which uniquely define the necessary weighting functions. The development and rationale behind the boundary value problems is the subject of the next section.

## IV. THE BOUNDARY VALUE PROBLEM

Having chosen equation (9) as the interpolation form for local representation of the function (7) in the intervals (10), complete definition of the weighting functions now depends on the choice of interpolation properties for equation (9). One set of reasonable properties is:

1) require the global function to reduce exactly to the local functional approximations at their centroids--not only in value but in their first m partial derivatives,

2) in the interior of an interval, the final functional model must be a "reasonable" weighted average of the local functions located at the endpoints of the interval. A reasonable average is obtained by requiring the neighboring weighting functions to add to unity for all values of the independent variable in the closed interval between their centroids. This condition will insure that if both local functional approximations defining an interval have the same value at some point in the interior of the interval, then the final function will also have that value, clearly a desirable property.

The conditions on the final functional model which result from the above interpolation properties are given in Table 1. Noting the symmetry in the conditions in Table 1, it follows that only one weighting function needs to be determined--all the others are simple translations, scalings and/or reflections of the first. The boundary value equations for only $J_1(x)$, centered at $x = 1$, valid on the interval from zero to one are given in Table 2. For example, $J_0(x)$, the weighting function centered at x = 0, also valid over the interval zero to one can be formed from $J_1(x)$ by

$$J_o(x) = J_1(1 - x) = 1 - J_1(x) \tag{11}$$

because of the symmetry of boundary conditions at either end of the interval. The equations in Table 2 follow directly from the required properties of equation (9) listed in Table 1. Solution of the boundary value problem for the required weighting function is the subject of the next section.

## V. THE WEIGHTING FUNCTIONS

The boundary value problem summarized in Table 2 can be alternatively stated as follows:

1) the first derivative of the weighting function must have an m-th order zero at the centroid of its respective local approximation,

2) the weighting function must have an (m + 1)th order zero at the centroid of its neighboring local approximation,

3) the sum of the weighting function and its reflection about the centroid of its neighboring local approximation must be unity over the entire closed interval between the two adjacent local functions.

One family of solutions of this boundary value problem can be obtained by assuming that the weighting function is a polynomial in the independent variable x, e.g.

$$J_j(x) = \sum_{\ell} C_\ell x^\ell \tag{12}$$

with the upper limit on the summation as yet undefined. Restricting the discussion to $J_1(x)$ on the closed interval zero to one, then the first derivative of $J_1(x)$ with respect to x must be,

$$\frac{dJ_1(x)}{dx} = C_o x^m (1 - x)^m \tag{13}$$

to satisfy the condition that the first m derivatives vanish at x = 0 and x = 1. $C_o$ is, at this point, an arbitrary constant. Integration of (13) for the appropriate form for $J_1(x)$ gives:

$$J_1(x) = C_o \int_o^x \hat{x}^m (1 - \hat{x})^m d\hat{x} + C_1 \tag{14}$$

The remaining conditions on $J_1(x)$ at the endpoints of the interval will completely define the constants in equation (14). The remaining conditions at the endpoints are

$$\left.\begin{aligned} J_1(0) &= 0 \\ J_1(1) &= 1 \end{aligned}\right\} \tag{15}$$

Clearly, for the success of this approach, a non-trival solution of equation (14) subject to the conditions (15) must exist. For solution of this problem, equation (14) must be evaluated at the endpoints of the interval of interest, e.g.

$$J_1(0) = C_o \int_o^o \hat{x}^m (1 - \hat{x})^m d\hat{x} + C_1 = 0 \tag{16}$$

$$J_1(1) = C_0 \int_0^1 \hat{x}^m (1 - \hat{x})^m d\hat{x} + C_1 = 1 \tag{17}$$

Equation (16) gives the result, $C_1 = 0$. Equation (17) shows that the appropriate choice for $C_o$ is:

$$C_o = [\int_o^1 \hat{x}^m (1 - \hat{x})^m d\hat{x}]^{-1} = \frac{\Gamma(2m + 2)}{\Gamma^2(m + 1)} = \frac{(2m + 1)!}{(m!)^2} \tag{18}$$

where $\Gamma(m)$ is the gamma function

$$\Gamma(m) = (m - 1)! \tag{19}$$

for any m integer and greater than zero. The general form for $J_1(x)$ can now be written as:

$$J_1(x) = \frac{(2m + 1)!}{(m!)^2} \int_o^x \hat{x}^m (1 - \hat{x})^m d\hat{x} \tag{20}$$

The third condition of Table 2, that is necessary for reasonable interpolation in the interior of the interval, has not yet been considered. Usually, normalization would be necessary to insure the "reasonable interpolation" property, e.g. form normalized $\bar{J}_i(x)$ from $J_1(x)$ and $J_o(x)$ according to the formula:

$$\bar{J}_i(x) = \frac{J_i(x)}{J_o(x) + J_1(x)};\ i = 0, 1. \tag{21}$$

then clearly $\bar{J}_1(x) + \bar{J}_o(x) = 1$. Fortunately this normalization is unnecessary; the polynomial weighting functions derived here satisfy the reasonable interpolation property $\{J_1(x) + J_o(x) = 1\}$ without normalization. Proof of these statements follows from a consideration of $J_o(x)$. Note that the first derivative of $J_o(x)$ must satisfy exactly the same conditions as the first derivative of $J_1(x)$. Therefore equation (13) can be repeated for $J_o(x)$:

$$\frac{dJ_o(x)}{dx} = C_o x^m (1 - x)^m \tag{22}$$

the necessary conditions on $J_o(x)$ at the endpoints of the interval are:

$$J_o(0) = C_o \int_o^o \hat{x}^m (1 - \hat{x})^m d\hat{x} + C_1 = 1 \tag{23}$$

$$J_o(1) = C_o \int_o^1 \hat{x}^m (1 - \hat{x})^m d\hat{x} + C_1 = 0 \tag{24}$$

Equations (23) and (24) are simply solved, and the appropriate form for $J_o(x)$ is:

$$J_o(x) = 1 - \frac{(2m + 1)!}{(m!)^2} \int_o^x \hat{x}^m (1 - \hat{x})^m d\hat{x} \tag{25}$$

Consideration of equations (20) and (25) shows quite simply that:

$$J_o(x) + J_1(x) = 1 \tag{26}$$

for all x.

At this stage one other point remains, recall the symmetry arguments which resulted in:

$$J_o(x) = J_1(1 - x). \tag{27}$$

From examination of equations (20) and (25) it is not obvious that this is true. However, since the polynomial weighting functions are of degree 2m + 1, and they satisfy 2m + 2 conditions, they are unique--and therefore the two representations of $J_o(x)$ must be equivalent. (Existence and uniqueness follow simply from the derivation of the weighting function.)

Consideration of the expression (20) permits development of analytic expressions for the coefficients of the weighting functions. This development will be the subject of the next section.

## VI. COEFFICIENTS OF THE WEIGHTING FUNCTIONS

Utilization of equation (20) and the binomial expansion:

$$(x + c)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} c^r \tag{28}$$

for $0 \leq r \leq n$

with $\binom{n}{r} = \frac{n!}{(n - r)!r!}$

allows a series expansion of the integrand of equation (20) to

$$J_1(x) = \frac{(2m + 1)!}{(m!)^2} \int_o^x \sum_{r=0}^{m} \binom{m}{r} x^m \hat{x}^{m-r} (-1)^{m+r} d\hat{x}, \tag{29}$$

which can be rearranged and integrated term by term to yield:

$$J_1(x) = K \sum_{r=0}^{m} \beta_r x^{2m-r+1} \tag{30}$$

with;

$$K = \frac{(2m + 1)!(-1)^m}{(m!)^2}$$

$$\beta_r = \frac{(-1)^r \binom{m}{r}}{2m - r + 1}$$

The expressions for K and $\beta_r$ completely define the coefficients of the weighting functions for m-th order continuity, analytically. Table 3 contains a summary of the one-dimensional weighting functions, $J_1(x)$, for various order of continuity.

## VII. SUMMARY OF THE ONE-DIMENSIONAL CASE

The results presented thus far document a new sequential interpolation and approximation technique for mathematically modeling discretely measurable functions of one variable. These techniques use piecewise functions to define a globally valid m-th order continuous model by sequentially operating on small subsets of the original input data. The piecewise, sequential determination of the global modeling function allows continuous modeling of an arbitrarily large set of data--while still maintaining reasonable limits on the order of the interpolating and/or approximating functions. The total arbitrariness of the form of the local functional approximations permits a new level of flexibility in the local, and therefore global, modeling capability. The required weighting functions for this technique have been shown to be simple low order polynomials in the independent variable; and analytic expressions for their coefficients have been developed for arbitrary order continuity. The essence of the one-dimensional WIT procedure is demonstrated in Figure 1.

A most important advantage of WIT is that it generalizes fully to N-dimensions. Numerous sequential and non-sequential techniques exist for interpolation and/or approximation in one-dimension, most however do not result in practical algorithms, when generalized to higher-dimensions. Some techniques exist for continuous modeling in N-dimensions--but they result in functional forms that are so complex that they are of little value in modeling large data sets. This is not the case for WIT, it maintains its continuity properties and efficiency in N-dimensions; and is practical for large data sets.

Discussions of the generalizations of this technique to n-dimensions is the subject of the next sections of this paper.

## VIII. MODELING IN N-DIMENSIONS

After consideration of the one-dimensional case the generalization to N dimensions proceeds quite easily. One possible choice for the regions of validity for the final piecewise functions is N-dimensional hypercubes. Without loss of generality these hypercubes can be assumed to have sides of unit length. The centroids of the local functional approximations are then constrained to lie on a uniform N-dimensional grid. The appropriate interpolation form is,

$$\Phi(x_1, x_2, x_3, \ldots, x_n) =$$

$$\sum_{i_1=j_i}^{j_1+1} \sum_{i_2=j_2}^{j_2+1} \cdots \sum_{i_n=j_n}^{j_n+1} J_{i_1 i_2 \ldots i_n}(x_i, \ldots, x_n) F_{i_1 \ldots i_n}(x_1, \ldots, x_n) \tag{37}$$

valid over the hypercube defined by:

$$\bar{x}_{j_k} \le x_k \le \bar{x}_{j_{k+1}} \quad \text{for } k = 1, 2, 3, \ldots, n \tag{38}$$

where the points $(\bar{x}_{j_1}, \bar{x}_{j_2}, \bar{x}_{j_3}, \ldots, \bar{x}_{j_4})$ are the centroids of the local functional approximations,

$$F_{j_1 j_2 \ldots j_n}(x_1, \ldots, x_n) \tag{39}$$

The interpolation properties which are required for mth order continuity and reasonable interpolation are given in Table 4. Considering only the weighting function $J_{111\ldots1}(x_1, \ldots, x_n)$ centered at the point (1, 1, 1, ..., 1) valid over the hypercube given by

$$0 \le x_k \le 1 \quad k = 1, 2, 3, \ldots, n. \tag{40}$$

Then the appropriate boundary value problem based on the chosen interpolation form and properties is given in Table 5. The solution to the stated problem is:

$$J_{11\ldots 1}(x_1,\ldots,x_n) = J_1(x_1)J_1(x_2)\ldots J_1(x_n) = \prod_{i=1}^{N} J_1(x_i). \quad (41)$$

and the remaining $2^N - 1$ weighting functions are given by symmetry as:

$$J_{i_1 i_2 i_3 \ldots i_N}(x_1, x_2, \ldots, x_N) = \prod_{j=1}^{N} J_1(z_j) \quad (42)$$

where

$$z_j = \begin{cases} x_j \text{ if } i_j = 1 \\ 1 - x_j \text{ if } i_j = 0. \end{cases} \quad (43)$$

The one dimensional weight functions are tabulated in Table 3.

## IX. APPLICATION OF THE TECHNIQUE

A set of 167,680 topographic elevation measurements was supplied by the U. S. Army Engineer Topographic Laboratories, Fort Belvoir, Virginia. The data ranged from 300 to 800 meters in elevation and covered approximately 75 square miles. The weighting function techniques developed were utilized to mathematically model the terrain data. The data was stored in functional form using only 5,541 coefficients. This resulted in reducing the physical storage required to only 3.3% of that needed for the original data. The functional model and weighting function theory was used as the basis of an automated contouring program and a contour map comprised of 50 contour levels was completed in only 3 minutes of CDC 6400 central processor time (Jancaitis and Junkins, 1973b). A three dimensional display of the area was completed in 4.4 minutes of CDC 6400 central processor time.

The results of this application demonstrates the ability of this technique for data compaction and efficient data analysis.

## X. CONCLUSION

The weighting function interpolation and approximation technique (WIT) for modeling discretely measurable functions of n independent variables with m-th order continuity has been developed. This technique utilizes locally valid piecewise functions formed from local functional approximations using polynomial weighting functions. The weighting functions have been formulated to insure that each piecewise locally valid function join with m-th order continuity with each of its adjacent piecewise functions. Analytic formula for the coefficients of the weighting functions for n-dimensions and m-th order continuity have been derived. This modeling technique is applicable to arbitrarily large data sets and allows complete freedom in the choice of the form of the local functional approximations.

Utilization of this technique for representation of large geodetic data sets typically results in very significant reductions in physical storage requirements and greater ease and flexibility of analysis of the data.

## REFERENCES

Davis, P. J., Interpolation and Approximation, Blaisdell Publishing Co., Waltham, Massachusetts, 1963.

Greville, T. N. E., Theory and Applications of Spline Functions, Academic Press, 1969, 212 pages.

Jancaitis, J. and J. Junkins, "Mathematical Techniques for Automated Cartography," Final Report on Contract No. DAAK02-72-C-0256 issued by U. S. Army Mobility Equipment Research and Development Center, Fort Belvoir, Virginia, February 1973b, 108 pages.

Jancaitis, J. R. and J. L. Junkins, "Modeling Irregular Surfaces," Photogrammetric Engineering, April, 1973a, pp. 413-420.

Junkins, J. L. and J. R. Jancaitis, "A Weighting Function Approach to the Modeling of Geodetic Surfaces," Transactions of the American Geophysical Union, Vol. 53, No. 4, 1972b, p. 346.

Junkins, J., Jancaitis, J. and G. Miller, "Smooth Irregular Curves," Photogrammetric Engineering, June 1972a, pp. 565-573.

Lee, W. H. K. and W. M. Kaula, "A Spherical Harmonic Analysis of the Earth's Topography," J. Geophysical Research, 72, 753, 1967.

Lundquist, C. A. and G. E. O. Giacaglia, "A Geopotential Representation with Sampling Functions," paper presented at the Symposium on Use of Artificial Satellites for Geodesy, American Geophysical Union, Washington, D. C., April 1971.

Morrison, F., "Density Layer Models for the Geopotential," American Scientist, Vol. 60, No. 2, March-April 1972, pp. 229-236.

Table 1. Interpolation Conditions on $\bar{\bar{x}}$ in the Interval $(\bar{x}_j \le x \le \bar{x}_{j+1})$ at $x = \bar{x}_j$.

$$\phi(\bar{x}_j) = F_j(\bar{x}_j)$$

$$\frac{d^K \phi(\bar{x}_j)}{dx^K} = \frac{d^K F_j(\bar{x}_j)}{dx^K} \quad k = 1, 2, 3, \ldots, m$$

<u>at $x = \bar{x}_{j+1}$:</u>

$$\phi(\bar{x}_{j+1}) = F_{j+1}(\bar{x}_{j+1})$$

$$\frac{d^K \phi(\bar{x}_{j+1})}{dx^K} = \frac{d^K F_{j+1}(\bar{x}_{j+1})}{dx^K} \quad k = 1, 2, 3, \ldots, m$$

<u>for all x, $\bar{x}_j \le x \le \bar{x}_{j+1}$:</u>

$$\phi(x) = J_j(x)F_j(x) + J_{j+1}(x)F_{j+1}(x)$$

$$J_j(x) + J_{j+1}(x) = 1$$

Table 2. Boundary Value Equations for $J_1(x)$ on the Interval $0 \le x \le 1$ $J_1(x)$ Centered at $x = 1$

<u>at $x = 0$:</u>

$$J_1(0) = 0$$

$$\frac{d^K J_1(0)}{dx^K} = 0 \quad k = 1, 2, 3, \ldots, m$$

<u>at $x = 1$:</u>

$$J_1(1) = 1$$

$$\frac{d^K J_1(1)}{dx^K} = 0 \quad k = 1, 2, 3, \ldots, m$$

<u>for all x, $0 \le x \le 1$:</u>

$$J_1(x) + J_0(x) = 1$$

or, equivalently

$$J_1(x) + J_1(1 - x) = 1$$

Table 3. One-Dimensional Weighting Function Coefficients

| Order of Continuity m | Coefficients of the Weighting Function (descending powers of x, from 2m+1 to m+1) | $J_1(x)$ |
|---|---|---|
| 0 | 1 | $x(1)$ |
| 1 | -2,3 | $x^2(-2x+3)$ |
| 2 | 6,-15,10 | $x^3(6x^2-15x+10)$ |
| 3 | -20,70,-84,35 | $x^4(-20x^3+70x^2-84x+35)$ |
| 4 | 70,-315,540,-420,126 | . |
| 5 | -252,1386,-3080,3465,-1980,462 | . |
| 6 | 924,-6006,16380,-24024,20020,-9009,1716 | . |
| 7 | -3432,25740,-83160,150150,-163800,108108,-40040,6435 | . |
| 8 | 12870,-109395,408408,-875160,1178100,-1021020,556920,-175032,24310 | |

$$J_1(x) = \frac{(2m+1)!(-1)^m}{(m!)^2} \sum_{r=0}^{m} \frac{(-1)^r \binom{m}{r}}{2m - r + 1} x^{2m-r+1}$$

Table 4. Interpolation Properties for N-Dimensional Modeling

at $X = (\bar{x}_{j_1}, \bar{x}_{j_2}, \ldots, \bar{x}_{j_n})$, (the centroids of the local approximations)

$$\left.\frac{\partial^{\ell_1+\ell_2+\ldots+\ell_n}\phi}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\ldots\partial x_n^{\ell_n}}\right|_X = \left.\frac{\partial^{\ell_1+\ell_2\ldots+\ell_n}F_{j_1 j_2\ldots j_n}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\ldots\partial x_n^{\ell_n}}\right|_X$$

$$\ell_i = 0, 1, 2, \ldots, m; \quad i = 1, 2, 3, \ldots, n$$

along $X_k(x_1, x_2, \ldots, x_{k-1}, \bar{x}_{j_k}, x_k, \ldots, x_n)$, (the boundary surfaces)

$$\left.\frac{\partial^{\ell_1+\ell_2+\ldots+\ell_n}\phi}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\ldots\partial x_n^{\ell_n}}\right|_{X_k} = \frac{\partial^{\ell_1+\ell_2+\ldots+\ell_n}}{\partial x_1^{\ell_1}\ldots\partial x_n^{\ell_n}}\left[\sum_{i_1=j_1}^{j_1+1}\cdots\sum_{i_{k-1}=j_{k-1}}^{j_{k-1}+1}\sum_{i_{k+1}=j_{k+1}}^{j_{k+1}+1}\cdots\right.$$

$$\left.\cdots\sum_{i_n=j_n}^{j_n+1}\left(J_{i_1\ldots j_k\ldots i_n}(x_1, \ldots, \bar{x}_{j_k}, \ldots, x_n)F_{i_1\ldots j_k\ldots i_n}(x_1, \ldots \bar{x}_{j_k}, \ldots, x_n)\right)\right]_{X_n}$$

in the closed hypercube $\bar{x}_{j_k} \le x_k \le \bar{x}_{j_{k+1}}$; $k = 1, 2, 3, \ldots, m$

$$\phi(x_1, x_2, \ldots, x_n) = \sum_{i_1+j_1}^{j_1+1}\sum_{i_2=j_2}^{j_n+1}\cdots\sum_{i_n=j_n}^{j_n+1} J_{i_1 i_2\ldots i_n}(x_1, x_2, \ldots, x_n)F_{i_1 i_2\ldots i_n}(x_1, x_2, \ldots, x_n)$$

with

$$\sum_{i_1=j_1}^{j_1+1}\sum_{i_2=j_2}^{j_2+1}\cdots\sum_{i_n=j_n}^{j_n+1} J_{i_1 i_2 i_3\ldots i_n}(x_1, x_2, \ldots, x_n) = 1$$

Table 5. Boundary Value Problem for
$J_{111\ldots 1}(x_1, x_2, \ldots, x_n)$

at $X = (1, 1, 1, \ldots, 1)$:

$$\left.\frac{\partial^{\ell_1+\ell_2+\ell_3+\ell_4\ldots+\ell_n}J_{111\ldots 1}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\ldots\partial x_n^{\ell_n}}\right|_X = 0$$

$$\ell_i = 0, 1, 2, 3, \ldots, m; \; i = 1, 2, 3, \ldots, N; \; \sum \ell_i \ne 0$$

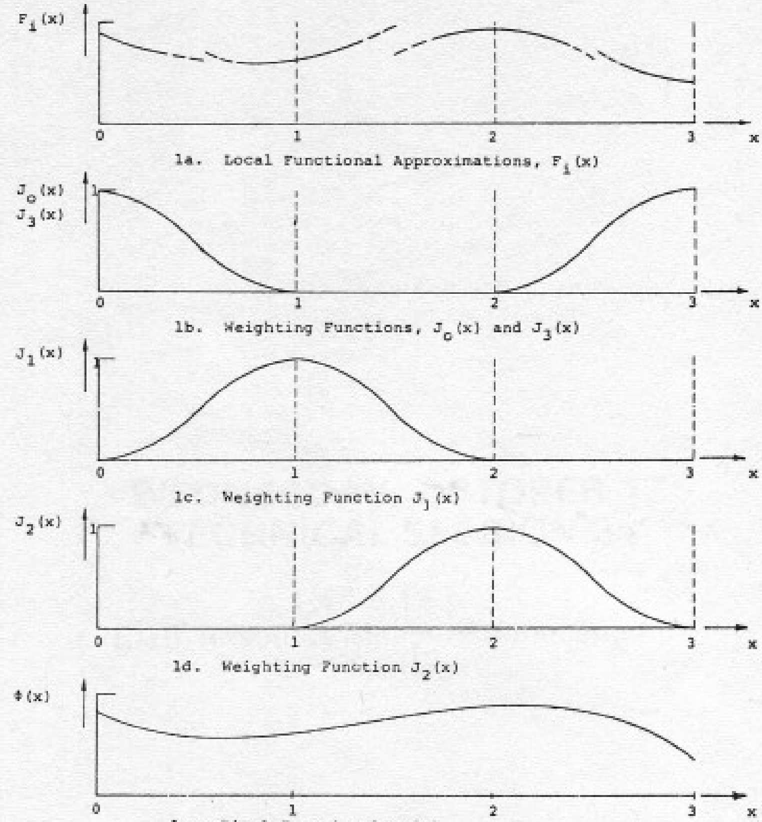
along $X_k = (x_1, x_2, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n)$:

$$\left.\frac{\partial^{\ell_1+\ell_2+\ell_3+\ldots+\ell_n}J_{111\ldots 1}}{\partial x_1^{\ell_1}\partial x_2^{\ell_2}\ldots\partial x_n^{\ell_n}}\right|_{X_k} = 0$$

$$\ell_i = 0, 1, 2, 3, \ldots, m; \; i = 1, 2, 3, \ldots, n$$

in the closed hypercube defined by $0 \le x_k \le 1$; $k = 1, 2, 3, \ldots, n$:

$$\sum_{i_1=0}^{1}\sum_{i_2=0}^{1}\sum_{i_3=0}^{1}\cdots\sum_{i_n=0}^{1} J_{i_1 i_2 i_3 i_4\ldots i_n}(x_1, x_2, x_3, \ldots, x_n) = 1$$




1a. Local Functional Approximations, $F_i(x)$

1b. Weighting Functions, $J_o(x)$ and $J_3(x)$

1c. Weighting Function $J_1(x)$

1d. Weighting Function $J_2(x)$

1e. Final Functional Model

$$\Phi(x) = J_i(x)F_i(x) + J_{i+1}(x)F_{i+1}(x)$$

for x; $i \leq x \leq i+1$

Figure 1. Weighting Function Interpolation Technique (WIT)