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LINEAR DIMENSIONALITY OF LANDSAT AGRICULTURAL DATA WITH

IMPLICATIONS FOR CLASSIFICATION

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I. ABSTRACT

A model for the LANDSAT multispectral scanner data, representing a generalization of the commonly used Gaussian model, has been formulated and analyzed. The model hypothesizes that the data for different crop types essentially lie on distinct hyperplanes in the feature space. Tests of this model reveal that: (1) the agricultural data from any single acquisition (i.e., four-channel) of LANDSAT are essentially two dimensional, regardless of the crop type; and (2) the data from different sites and different stages of crop development all lie on planes which are parallel. These findings have significant implications for data display, classification, feature extraction, and signature extension.

II. INTRODUCTION

Standard models used in classifying remotely-sensed multispectral data from agricultural sites start with an assumption that observations from a field constitute a sample from a probability distribution characterized by its crop-type. In most cases this probability distribution is assumed to be Gaussian, completely specified by a mean vector and a dispersion matrix. Estimates of these parameters based on a sample characterize the crop-type, and are called its "signature." This model will be referred to as the point-signature model.

Experience with LANDSAT multispectral scanner data, however, has shown that the average values of observations from different agricultural fields of the same crop usually vary much more than would be expected under the assumption of a common probability distribution. As an example, Figures 1 and 2 provide plots of mean vectors for several wheat and nonwheat fields in a LANDSAT subframe. These data were collected in four different passes of the Satellite over the same location at different times during the growth cycle of wheat, each pass contributing a 4-vector of observations. The idea of using data from multiple acquisitions at different biological phases of the wheat crop to estimate wheat acreage in the United States is currently being used in an experiment being conducted by NASA[1]. In Figures 1 and 2, the variability of data within fields has been indicated by bars of length two standard deviations for some representative fields and channels. As can be seen, the variability of the mean values is much larger than would be expected from the point-signature model. In fact, hypothesis tests for equality of mean vectors across fields usually fail even if the pixel-to-pixel correlation structure of the multispectral scanner data be taken into account.

The objective of the work presented in this paper is to analyze a model for the data which generalizes the point-signature model by permitting a limited variability in the probability distributions associated with the different fields of a class. The hope is that use of such a model will aid with our understanding of LANDSAT data and lead to improvements in our ability to classify these data. The model is described in the next section, and, with greater mathematical detail, in the Appendix. Section IV presents the results of tests of this model with the LANDSAT data from single and multiple acquisitions, and discusses their implications for data display and classification.

III. THE HYPERPLANE MODEL

The model is described by the following testable assumption: The mean values associated with different fields of a crop-type all lie on a single hyperplane of dimension r, which can be determined from the data. That is, the mean vector μ_{ij} for the j-th field in class i has the form

$$\mu_{ij} = \mu_{oi} + E_i \alpha_{ij} \qquad (1)$$

where μ_{oi} and E_i are, respectively, a p-vector and a p×r matrix where p (>r) is the number of channels used in gathering the data. Under this model, the hyperplane associated with the i-th class is represented by the pair (μ_{oi}, E_i) and the r-vectors α_{ij} account for the variability observed in the field averages. The use of the model, if it is shown to hold for LANDSAT data, lies in the fact that it constrains the observed variability of field mean values to lie within a space of lower dimensionality than that represented by the number of channels. The remaining dimensions are either noise or represent changes in the structure of the hyperplanes across classes.

A likelihood ratio test for hypothesis (1) and the maximum likelihood estimates for μ_{ij}, μ_{oi}, E_i, and α_{ij} are given in the Appendix. The derivation of the test statistic, used to establish dimensionality r, and the maximum likelihood estimates of the parameters have assumed that the data are multivariate normal, independent from pixel to pixel, and have a common, known covariance matrix. Since these assumptions are generally violated for LANDSAT data, the distribution of the test statistic, shown to be chi-squared in the Appendix, is only approximately chi-squared.

IV. TESTING THE MODEL ON LANDSAT DATA

Single-Pass Data

The model was tested on LANDSAT data from several agricultural sites with identical results. As an example, results are reproduced here for four-channel data from a site in Nebraska. These data were taken from a set of known fields: ten of wheat and eleven of nonwheat. Field averages for some of these are given in Table 2 and plotted in Figure 1. The model (1) was tested on data for each set of fields separately, and then on all fields taken together. The test statistics and the estimates of the parameters of the plane are given in Table 1.

The test statistic measures the total deviation of the mean vectors of the fields in the class from the hypothesized r-dimensional plane (r=0,1,2,3). Suppose data are given for k_i fields of class (i.e., crop type) i. Let n_{ij} and $\bar{x}_{ij}$ be, respectively, the number of pixels and the average value of the observations for the j-th field, $j=1,2,\ldots,k_i$, and let $\hat{\Sigma}_i$ be the estimated covariance matrix for the fields of this class. The values of the test statistic for different values of r are given by

$$t_{ir} = \sum_{j=1}^{k_i} n_{ij} \, (\bar{x}_{ij}-\hat{\mu}_{ij;r})' \, \hat{\Sigma}_i^{-1} \, (\bar{x}_{ij}-\hat{\mu}_{ij;r}),$$

where $\hat{\mu}_{ij;r}$ is the best estimate of μ_{ij} on an r-dimensional plane which gives the best possible fit to the data. For example, $\hat{\mu}_{ij;o}$ is the average (over j) of $\bar{x}_{ij}$; $\mu_{ij;1}$ is the best estimate of the mean value on the straight line of best fit; and so on.

For each of the three classes (nonwheat, wheat, and pooled data) in Table 1, there is a sharp drop in the values of the test statistic between r=1 and r=2. Note that the values of the test statistics for r=2 in each case are large when compared with a chi-squared distribution with the appropriate degrees of freedom. This, however, is to be expected in view of earlier remarks about the violation of assumptions under which the distribution of the test statistic was derived. The conclusion from Table 1 is that these data (nonwheat and wheat) all lie on a two-dimensional plane.

The plane of data variability for each of the three classes in Table 1 has been characterized by $\hat{\mu}_o$ and the first two of the orthonormal basis vectors, e_1 and e_2. $\hat{\mu}_o$ is a point on this plane, and e_1 and e_2 span a subspace in which the vectors {observation - $\hat{\mu}_o$} essentially lie. In spite of the different look each characterization appears to have, the planes so defined are nearly identical. For example (using superscripts nw and w to denote identification of a vector with classes nonwheat and wheat, respectively), it is easily seen that $\{e_1^{nw}, e_2^{nw}\}$ and $\{e_1^{w}, e_2^{w}\}$ span the same subspace; the component of e_1^{nw} along e_1^{w} is 0.997, and its total component in the subspace spanned by $\{e_1^{w}, e_2^{w}\}$ is 0.9995.

To complement this geometric argument, we have developed a chi-squared test for the "equality" of planes. Using this approach, the test statistic for the pooled data is partitioned into a sum of components, which provide tests for dimensionality, parallelness, and equality of planes of the individual classes. Use of this technique confirms the conclusion from Table 1 that the planes determined by the nonwheat and wheat data separately, are identical.

A similar analysis of data from several sites has led to the same conclusion; namely, the four-channel LANDSAT data lie on a two-dimensional plane, and the planes associated with data from different sites and different stages of crop development are all essentially parallel. This fact has very significant implications for data display, feature extraction, classification, and signature extension.

Consider the following transformation of the observed field means $\{\bar{x}_i\}$.

$$\bar{y}_i = \left[e_1 \;\vdots\; e_2 \;\vdots\; e_3 \;\vdots\; e_4\right]' \; \bar{x}_i$$

The transformation matrix is orthogonal and vector $\bar{y}_i$ gives the components of the observed field mean in a new coordinate frame obtained by a rotation of the old one. The advantage of this representation of observations lies in the fact that the first two axes of the new coordinate frame are parallel to the plane of data variability and the other two are orthogonal to it. The first two elements of $\bar{y}_i$ give the components of the observation vector in the plane of data variability, and each of the remaining two is nearly the same regardless of the class from which the observation came.

The observed mean vectors for wheat and nonwheat fields given in Table 2 (see also Figure 1) were transformed to get their representation in the rotated coordinate frame. As a demonstration of the near-parallelness of the planes of variability from one set of data to another, the orthogonal matrix used here is the one obtained from the data for 16 nonwheat fields from a site in Oklahoma. The original and transformed observations are given in Table 3.

In the new coordinate frame, the relative positions of the data are almost entirely characterized by the first two components. A plot of these two components on a plane retains the relative positions of the data points as they were in the original observations, and reveals the separability (or, confusion) among wheat and nonwheat data not apparent in the table of the original data.

Figures 3a through 3d give plots of the components of the means of the nonwheat and wheat fields in the plane of data variability for data acquired in four different passes of LANDSAT over this Nebraska site at times chosen to coincide with distinct biological growth phases of the wheat crop. The data shown in Table 3 are among those plotted in Figure 2a. For each of the four sets of data corresponding to four passes, the plane of data variability used is the one determined for the Oklahoma site referred to earlier. It should be noted that variability seen in the last two components of the transformed means would have been considerably less had we used in each case the plane determined by the data from that pass. The plane, after all, is estimated by "best" fit to data. The fact that a plane that best fitted data from a set of 16 nonwheat fields in Oklahoma at a certain time of the year can reasonably fit the data from wheat and nonwheat fields in Nebraska collected at four different times of the year appears to be fairly strong evidence for constancy (or, at least, parallellness) of the planes of data variability.

Now consider the matter of classification of data. The original hypothesis was that the wheat data lie on a distinct plane which might be characterized as its signature. This loosening up of the standard point-signature model, it was hoped, would account for much of the variability in wheat data by disregarding the position of a data point on the plane and taking the classification statistic as the distance from the plane. For data acquired in a single pass of LANDSAT, it was seen that no distinction can be made in the planes of variability for the wheat and the nonwheat data. The position of the data points on the plane, however, can be used to discriminate among classes. Figures 3a through 3d, plots of the components of wheat and nonwheat field averages in the plane of variability, show a tendency for the elements of a class to group or cluster. Since the location of points in the plane is governed by the α vectors in the basic model (1), discrimination can be based upon estimates of these values. The maximum likelihood estimate of the α associated with observation x belonging to the i-th class is given by

$$\hat{\alpha} = (E' \Sigma_i^{-1} E)^{-1} E' \Sigma_i^{-1} (x - \hat{\mu}_{oi})$$

and, asymptotically, $\hat{\alpha}$ has a r-variate normal distribution with mean, say $\bar{\alpha}_i$, which depends upon the class and covariance matrix given by the expression

$$(E' \Sigma_i^{-1} E)^{-1}$$

Thus, a maximum likelihood classification procedure in the plane is to estimate the α vectors from training fields for a class, estimate their mean value $\bar{\alpha}_i$, and classify by minimizing the statistics

$$H_i = (\hat{\alpha} - \bar{\alpha}_i)^1 E' \Sigma_i^{-1} E (\alpha - \bar{\alpha}_i)$$

over the choice of classes. This is essentially a linear feature selection technique which reduces the dimensionality by disregarding the noise contributed by components outside the variability plane. Tests of this technique have yielded classification results virtually identical to results achieved using the original four-dimensional data. The prime advantage of this technique for LANDSAT data lies in the fact that classification results can be compared visually with a plot in two dimensions.

Multiple-Pass Data

Test of model (1) with multiple-acquisition data was also carried out for data from several sites. Results of this part of the study are not conclusive because it was difficult to find enough data from locations having multiple passes and a relatively large number of defined fields in the different classes to perform a satisfactory analysis. The tentative conclusion is as follows: While four-channel data from the different acquisitions taken separately lie on parallel planes, taken together the wheat and nonwheat data tend to lie on hyperplanes whose distinctness becomes more discernible as the number of passes increases from two to four. There, however, are some unresolved issues.

Results are presented here for four-pass registered data from a site in Kansas. Data were taken for a set of 42 known fields: 28 of wheat and 14 of fallow. The wheat fields were arbitrarily assigned to one of two wheat classes so that each contained 14 fields. The fallow fields constituted class 3. The likelihood ratio test statistics for dimensionality analysis were computed for each class separately, and for class "ALL" consisting of all 42 fields. The likelihood ratio test statistics for class wheat 1 are given in Table 4. The data for each of the three classes were found to lie essentially on eight-dimensional hyperplanes; the pooled data for the three classes appear to lie on an 11- to 12-dimensional hyperplane.

The fact that the pooled data lie on a hyperplane whose dimensionality is greater than that for the data for each class taken separately suggests distinctness of the class hyperplanes. To determine the relative orientations of the eight-dimensional class hyperplanes in the feature space, the component of each basis vector for the hyperplane of a class was computed in the subspaces spanned by the eight basis vectors of each of the other two classes. Clearly, if the hyperplanes were parallel, each basis vector for the hyperplane of a class would be entirely contained in the subspace spanned by the eight basis vectors for every other class. The first seven basis vectors of the wheat 2 hyperplane were found to have components of length 0.937-0.981 in the subspace spanned by the basis vectors of wheat 1; the eighth basis vector, however, had a large component (≈0.8) out of this subspace. Only the first five basis vectors of the fallow hyperplane had large components parallel to wheat 1 hyperplane. The inadequacy of data (an eight-dimensional hyperplane fitted to 14 points in a 16-dimensional space) makes it difficult to draw definitive conclusions, but it appears that the wheat 1 and wheat 2 hyperplanes are nearly parallel, while the wheat and fallow hyperplanes are not. This, though, is difficult to reconcile with the behavior of the four-channel data analyzed separately for each pass.

It is instructive to examine how well the wheat 1 hyperplane fits the data from wheat 2. To this end, maximum-likelihood estimates of the mean vectors of fields in class wheat 2 are computed on the hyperplane of wheat 1. Table 5 gives the observed mean vectors and the estimates for three of the fields. In view of the large differences in the observed mean vectors, the fit is surprisingly good.

The classification of data with the model of distinct hyperplanes entails computation of the following weighted distances of the observation from the different class hyperplanes

$$d_i = (x - \hat{\mu}_i)' \; \Sigma_i^{-1} \; (x - \hat{\mu}_i) \; ,$$

where x is the observation, $\hat{\mu}_i$ its estimate on the hyperplane of class i, and Σ_i the covariance matrix for the fields in class i. The observation is assigned to a class for which this distance is the shortest. The per-field classification of wheat 2 data in wheat 1 and fallow using this procedure assigned the data to wheat 1 with no misclassification.

V. DISCUSSION

The test of the proposed model for LANDSAT multispectral scanner data from agricultural fields has yielded useful information on the structure of these data. The two-dimensional representation of the four-channel data by a known transformation provides a valuable tool for data analysts. Implications of this structure of the data for feature extraction and classification have been discussed in Section IV. An additional aspect of crop classification using LANDSAT data is the so-called signature extension problem, which consists of estimating mean observation vectors for a crop type at a site on the basis of training data available at another site. Clearly, the fact that the data must lie on a known plane provides an important constraint for this estimation procedure.

Preliminary analysis suggests that the conclusion on parallelness of the planes of variability of data from different acquisitions may be strengthened--the planes may be identical. Additional work, however, is needed to establish this. Experience with a limited amount of nonagricultural data (water, mountains, roads, clouds, etc.) indicates that these too may lie in the same plane as the agricultural data, but, here again, additional work is required.

Tests of the proposed model using data from four acquisitions during different biological phases of the wheat crop reveal that the data, treated as a 16-dimensional vector, essentially lie on an eight-dimensional hyperplane. Also, the hyperplaness defined from multiple passes appear to be sufficiently separated to allow classification based on distance from the class hyperplanes. This distinction of the multiple pass hyperplanes, if not an artifice of our limited data set, appears to be inconsistent with the results from dimensionality analysis of single pass datam, where the class planes were found indistinguishable.

Perhaps a physical interpretation can be given to the plane of variability of the agricultural data. In a model developed by Kauth[2,3] an attempt was made to extract from the data the variability attributable only to changes in the soil type. This was done by identifying in the feature space a plane on which most of the variability of the soil reflectances lay. The information on the green and yellow crop development, it was concluded, lay in the directions orthogonal to the plane. The result was the so-called Tasselled Cap coordinate frame where components of the transformed observation were identified with soil, growing vegetation, mature vegetation, and noise. In the transformations presented in this paper, however, no such identification of directions or planes could be made with the crop phenology.

APPENDIX

Linear Functional Relationship Among Mean Vectors

To study the nature of variability among the mean vectors of fields of a certain class, consider the following model. Let there be p-variate normal populations $N_p\ (\mu_1, \Sigma), \ldots, N_p\ (\mu_k, \Sigma)$ with a common dispersion matrix. Consider the following hypothesis on the functional relationship among the population means

$$\mu_i = \mu_o + E\,\alpha_i, \quad i = 1, 2, \ldots, k; \; k > p, \qquad (A.1)$$

where μ_o is an unknown p-vector, E is an unknown p×r matrix of full rank (r<p), and α_i is an unknown r-vector. According to hypothesis (A.1), the population means $\{\mu_i\}$ lie on an r-dimensional hyperplane completely defined by μ_o and E. Given the sample means from these populations, a hypothesis test can be carried out to see if (A.1) holds for some value of r and, if so, to identify the hyperplane by estimating μ_o and E. Note that μ_o is a point on the hyperplane, and, speaking loosely, the column vectors of E span the hyperplane. The matrix E is not uniquely defined, since the mean value μ_i does not

change if E is replaced by EA, where A is nonsingular, and α is replaced by $A^{-1}\alpha$.

Consider two special cases of (A.1): If $\mu_1 = \mu_2 = \ldots = \mu_k$, it would be concluded that these population means lie on a zero-dimensional hyperplane; i.e., r = 0; conversely, if the $\{\mu_i\}$ are located randomly in the p-dimensional space, hypothesis (A.1) would be rejected for all values of $r < p$.

Suppose that n_i observations are taken from $N_p(\mu_i, \Sigma)$ and $\bar{x}_i$ is their average value (i = 1, 2, ..., k). Let $\bar{x}$ be the overall average value and B the between groups corrected sum of squares and products matrix:

$$\bar{x} = \sum_{i=1}^{k} n_i \bar{x}_i \Big/ \sum_{i=1}^{k} n_i$$

$$B = \sum_{i=1}^{k} n_i (\bar{x}_i - \bar{x})(\bar{x}_i - \bar{x})'.$$

The logarithm of the likelihood of the observed sample means is then

$$t = -\tfrac{1}{2} \sum_{i=1}^{k} n_i (\bar{x}_i - \mu_i)' \Sigma^{-1} (\bar{x}_i - \mu_i) .$$

The test statistic for likelihood ratio test of hypothesis H_o (A.1) is

$$L = \min_{H_o} \sum_{i=1}^{k} n_i (\bar{x}_i - \mu_i)' \Sigma^{-1} (\bar{x}_i - \mu_i)$$

$$= \min_{\mu_o, E, \alpha_i} \sum_{i=1}^{k} n_i (\bar{x}_i - \mu_o - E\alpha_i)' \Sigma^{-1} (\bar{x}_i - \mu_o - E\alpha_i).$$

From References [4] and [5], minimization of the expression on the right-hand side yields

$$L = \lambda_{r+1} + \ldots + \lambda_p ,$$

where $\{\lambda_i\}$ are the roots, arranged in decreasing order, of determinantal equation

$$|B - \lambda\Sigma| = 0,$$

and under the null hypothesis, L follows a chi-squared distribution with (p-r) (k-r-1) degrees of freedom

$$L \sim \chi^2 \{(p-r)(k-r-1)\}.$$

The likelihood ratio test for dimensionality consists of computing $\{\lambda_i\}$ and comparing $\eta_m = \sum_{j=m+1}^{p} \lambda_j$, m = 0, 1, ..., p-1, with the distribution function of a χ^2 with $\{(p-m)(k-m-1)\}$ degrees of freedom. If η_r and η_{r+1} are significantly small when compared to their χ^2 distribution while η_{r-1} and η_{r-2} are large, then the dimensionality of the configuration of the mean vectors is inferred to be r.

The above analysis assumes that the k normal populations have a common dispersion matrix Σ, which is known. If Σ is not known, an estimate can be substituted in its place. The χ^2 test remains a valid asymptotic test, though approximate for finite sample sizes. The assumption of equality of the dispersion matrices is usually justifiable on the grounds that the tests on means are, as a rule, sufficiently robust against this violation. The computation of the test statistic, without this assumption, requires difficult numerical minimization.

To determine the maximum likelihood estimates of the parameters $\{\mu_o, E\}$ of the hyperplane, consider the following simultaneous reduction of matrixes Σ and B. There exists a nonsingular matrix M such that

$$\Sigma = MM'$$

and

$$B = M \Lambda M'$$

where

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p).$$

Then, from Reference [5],

$$\hat{E} = M \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

consists of the first r columns of M, denoted later as M_1, and $\hat{\mu}_o = \bar{x}$.

The maximum likelihood estimate of μ_i is given by:

$$\hat{\mu}_i = \bar{x} + M_1 (M^{-1})^1 (\bar{x}_i - \bar{x}),$$

where $(M^{-1})^1$ consists of the first r rows of M^{-1}. Because E in the basic model is not unique, M_1 has frequently been replaced in our discussion by an orthonormal matrix E derived from M_1. This leads to some simplification in discussing the planes. For general E, the estimate of α_i is given by $\hat{\alpha}_i = (E' \Sigma^{-1} E)^{-1} E' \Sigma^{-1} (\bar{x}_i - \bar{x})$.

Note that the likelihood ratio test statistic η_r is the sum of generalized Euclidean distances between $\bar{x}_i$ and $\hat{\mu}_i$

$$\eta_r = \sum_{i=1}^{k} n_i (\bar{x}_i - \hat{\mu}_i)' \Sigma^{-1} (\bar{x}_i - \hat{\mu}_i),$$

and the r-hyperplane $\{\hat{\mu}_o, \hat{E}\}$ is obtained by a weighted least-squares fit through the configuration of $\{\bar{x}_i\}$, i = 1, 2, ..., k.

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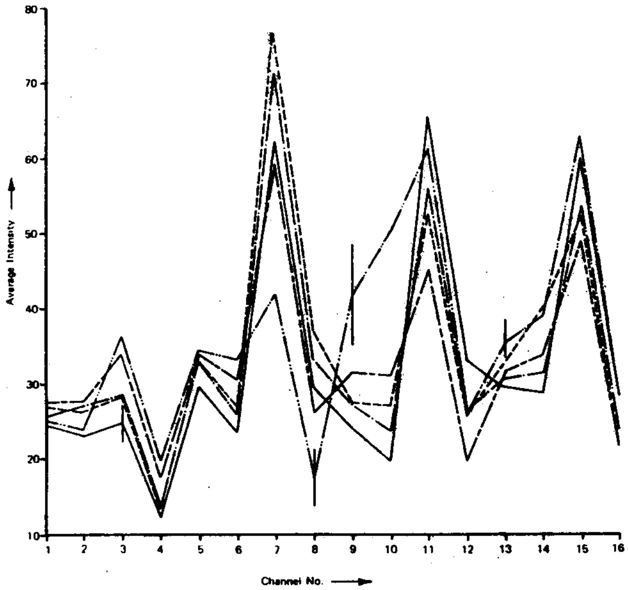

Figure 1. A Plot of Mean Vectors for Wheat Fields

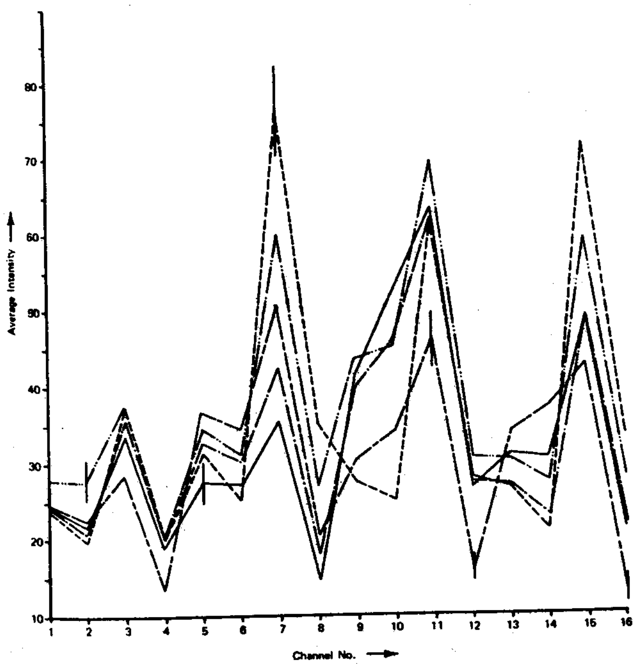

Figure 2. A Plot of Mean Vectors for Nonwheat Fields

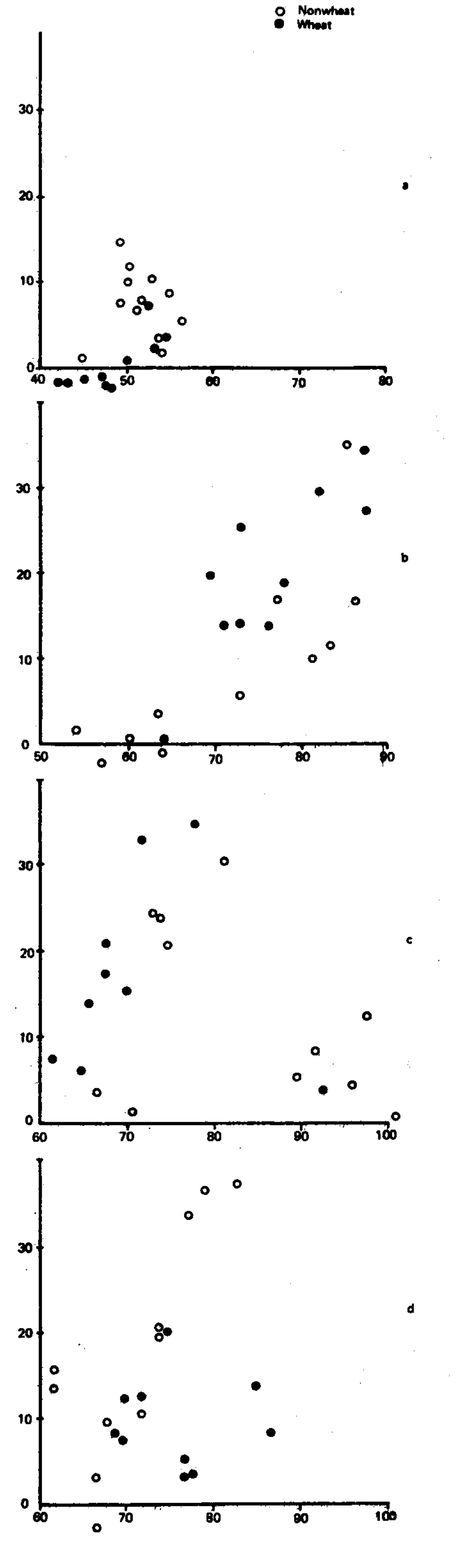

Figure 3. Plot Of Wheat And Nonwheat Data On The Plane Of Data Variability

Table 1. The Dimensionality Test Statistics and Estimates

	r	Degrees of Freedom	Test Statistic	$\hat{\mu}_o$	Orthonormal Basis Vectors e_1	e_2	e_3	e_4
1. Nonwheat data (11 fields)	0	40	1038.0	25.234	0.409	-0.072	0.909	-0.047
	1	27	335.9	23.512	0.856	-0.242	-0.393	0.227
	2	16	70.0	36.164	-0.077	-0.762	-0.059	-0.641
	3	7	15.5	20.309	-0.307	-0.595	0.129	0.732
2. Wheat data (10 fields)	0	36	1153.2	26.113	-0.144	-0.436	0.685	0.566
	1	24	274.4	25.465	-0.265	-0.818	-0.502	-0.091
	2	14	56.7	30.042	-0.813	0.129	0.306	-0.478
	3	6	11.4	15.266	-0.497	0.352	-0.431	0.666
3. Pooled data (21 fields)	0	84	3505.5	25.639	-0.081	-0.420	-0.780	0.457
	1	60	1393.6	24.412	-0.177	-0.861	0.477	-0.009
	2	38	155.8	33.343	0.764	0.283	0.237	0.530
	3	18	47.7	17.986	0.616	0.048	0.328	0.715

Table 2. Field Means and Their Maximum Likelihood Estimates on the Plane of Data Variability

FIELD #	NONWHEAT DATA Field Mean	NONWHEAT DATA Maximum Likelihood Estimate on the Plane	WHEAT DATA Field Mean	WHEAT DATA Maximum Likelihood Estimate on the Plane
1	24.421	24.398	22.500	24.863
	21.816	21.853	23.012	22.926
	34.053	33.937	24.817	25.096
	19.158	19.188	12.329	12.351
2	23.933	23.542	27.000	26.597
	19.867	20.057	26.233	26.458
	37.233	36.628	28.400	27.686
	21.633	21.707	13.100	13.205
3	24.212	23.965	26.981	26.735
	20.909	20.937	26.667	26.737
	35.894	35.799	28.222	27.994
	20.879	20.833	13.352	13.349
4	27.729	27.290	25.750	25.453
	27.708	37.753	23.854	23.998
	37.521	37.369	36.104	35.645
	20.042	19.955	19.750	19.806
5	24.700	24.819	27.455	27.236
	22.450	22.806	27.682	27.689
	28.450	27.342	33.841	33.811
	13.650	14.042	17.409	17.352

Table 3. Field Mean Vectors in the Original and Rotated Coordinated Frames

FIELD #	NONWHEAT DATA Field Mean	NONWHEAT DATA Transformed Field Mean	WHEAT DATA Field Mean	WHEAT DATA Transformed Field Mean
1	24.421	49.624	22.500	42.757
	21.816	7.437	23.012	1.807
	34.053	6.323	24.817	5.660
	19.158	6.408	12.329	5.854
2	23.933	50.910	27.000	48.203
	19.867	11.679	26.233	2.151
	37.233	7.102	28.400	6.068
	21.633	6.567	13.100	5.515
3	24.212	50.602	26.981	48.402
	20.909	9.900	26.667	2.335
	35.894	6.615	28.222	5.689
	20.879	6.740	13.352	5.772
4	27.729	56.955	25.750	52.854
	27.708	5.364	23.854	7.257
	37.521	5.664	36.104	6.325
	20.042	6.434	19.750	6.279
5	24.700	45.165	27.455	53.814
	22.450	1.062	27.682	2.112
	28.450	6.522	33.841	5.455
	13.650	5.129	17.409	6.177

Table 4. Likelihood Ratio Test Statistics for Dimensionality Analysis of 4-Pass (16-Channel) Data

Dimensionality	Degrees of Freedom	Statistics for Wheat 1 (14 Fields)
0	208	15,616.1
1	180	9,837.1
2	154	5,898.6
3	130	3,728.9
4	108	1,941.4
5	88	1,102.8
6	70	552.1
7	54	285.7
8	40	139.0
9	28	53.4
10	18	25.5
11	10	7.8
12	4	1.6
13	0	0.0
14	-	0.0
15	-	0.0

Table 5. The Mean Observation Vectors for Fields of Wheat 2 and Their Maximum-Likelihood Estimates (MLE) on the Wheat 1 Hyperplane.

	Field 1		Field 2		Field 3	
	Mean	MLE	Mean	MLE	Mean	MLE
1	32.949	32.870	37.291	37.491	34.845	34.630
2	35.424	35.107	42.400	41.881	38.000	37.866
3	35.848	36.051	46.418	46.285	47.108	47.153
4	17.071	16.972	22.945	22.859	24.101	24.056
5	26.808	27.311	28.236	28.390	26.081	26.695
6	28.232	28.696	30.691	30.278	26.507	26.713
7	31.838	32.607	36.945	36.607	41.838	41.527
8	16.555	16.595	19.654	19.501	24.405	24.422
9	40.293	40.785	41.927	41.375	40.905	41.875
10	46.040	46.432	49.200	48.242	48.790	48.477
11	52.859	53.760	54.909	54.025	55.074	54.604
12	26.091	25.995	26.709	26.391	26.993	26.876
13	52.040	51.351	52.018	52.788	54.108	54.200
14	63.434	61.481	61.273	61.466	64.182	62.917
15	67.566	66.242	66.073	67.956	74.115	74.578
16	31.727	31.461	31.273	31.587	34.561	34.627