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A METHOD OF INTERPOLATION IN TWO DIMENSIONS, AND ITS APPLICATION TO MULTI-SPECTRAL IMAGE REGISTRATION

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I. ABSTRACT

A method for interpolating a bidimensional surface through irregular sample points is proposed. The method includes the usage of finite element techniques and is applicable to multi-spectral image registration. A partition method to generate the grid over the image is also proposed.

II. INTRODUCTION

An interactive computer system for manipulation of multispectral images was developed at IIMAS (Instituto de Investigaciones en Matematicas Aplicada y Sistemas) for earth resources analysis. It is a set of programs called "SYSTEM PR" written in ALGOL for the B6700 computer, at present being transferred to a PDP 11-70.

The system's input is at the moment LANDSAT, SKYLAB or LARSYS images, being these the ones in use at IIMAS. Images can (1) be printed by the usual line printer or by an electrostatic printer, (2) be enhanced using several methods, (3) be classified using one of four different classifiers: maximum likelihood, bayesian, table look-up or clustering. The user is able to generate additional "bands" (gradient, ratio), or average images depending on the results desired. "PR" can be used either interactively or batch mode, allowing the user to supervise small tests through the remote terminal and make bulk processes in batch.

Several government agencies have been using the system for the last two years. DETENAL (Direccion de Estudios del Territorio Nacional), the mexican mapping agency, has involved IIMAS in a one year project which consists on generating land use maps automatically. Two areas, size

1,125,000 hs. each, were classified in 12 different classes using just one date image. Results were very good, although it is clear that two dates should be used in the future because of the confusion of some crops. As the rainy season image was used, grass and agriculture were sometimes confused, so it is advisable to use the dry season image also. This made necessary the implementation of registration procedures in order to use more information during classification.

C.F.E. (The Federal Electricity Comission), is another agency using the system. IIMAS and CFE are developing techniques in order to identify geothermal zones in the country using thermal Skylab and airborne scanner data, and heat models. Three areas in the volcanic axis and one in the North of Mexico will be studied, and the images obtained for several paths for day and night passes must be registered.

With these two projects, it became necessary to develope automatic registration techniques. The method used and the results obtained are presented below.

III. PROBLEM STATEMENT

Let A and B be two images covering the same area at different dates (fig.1). If pixels in image A are expressed in terms of (X,Y) and pixels in image B in terms of (U,V), let

(X_i, Y_i), $i=1,\ldots,m$

be m ground control points in A with (U_i, V_i) coordinates in B.

These m ground control points are found using an SSDA (Sequential Similarity Detection Algorithm) and a Monotonic-Increasing Threshold Algorithm[1].

The problem is how to find functions F and G such that

$F(X_i, Y_i) = U_i$
$G(X_i, Y_i) = V_i$ $i=1,\ldots,m$

and

$\int \int \ h(F(x,y)) + h(G(x,y)) \ dx \ dy$

taken over the whole image A is minimum, where h is the functional

$$h(z) = (\frac{\partial^2}{\partial x^2} z + \frac{\partial^2}{\partial y^2} z)^2 .$$

Functions F and G can be solved independently, so hereafter we will only be speaking of solving the problem for a single function

$Z(X_i, Y_i) = U_i$

IV. EXISTING METHODS

The problem has been solved either using a Green's Function or a Partial Differential Equation (P.D.E.). The first expresses $Z(x,y)$ as

$$Z(x,y) = (\sum_{i=1}^{m} a_i \, r_i(x,y)) + a_{m+1} + a_{m+2}x + a_{m+3}y$$

where

$$r_i(x,y) = \sqrt{(x-x_i)^2+(y-y_i)^2} \cdot \log\sqrt{(x-x_i)^2+(y-y_i)^2}$$

and

a_i, $i=1,..,m+3$ is a solution for $[C]\{a\}=\{r\}$, where $[C]$ is the symmetric matrix given by

$$C(i,j) = \begin{cases} r_i(x_j,y_j), & i=1,\ldots,m; \ j=1,\ldots,m \\ 1, & i=m-1; \ j=1,\ldots,m \\ & j=1,\ldots,m; \ i=m+1 \\ x_j, & i=m+2; \ j=1,\ldots,m \\ & i=1,\ldots,m; \ j=m+2 \\ y_j, & i=m+3; \ j=1,\ldots,m \\ & i=1,\ldots,m; \ j=m+3 \\ 0, & \text{elsewhere.} \end{cases}$$

and

$$r_i = \begin{cases} u_i, & i=1,\ldots,m \\ 0, & i=m+1,\ldots,m+3 \end{cases}$$

Matrix $[C]$ has rank m and therefore cannot be readily inverted. However, a variable transformation can be introduced:

$$\{a\}=[H]\{c\},$$

where $[H]$ is and $(m+3) \times m$ banded matrix of bandwith 3, and the system

$$[H]'[C][H]\{c\} = [H]'\{r\}$$

can be solved for $\{c\}$.

This type of solution has three drawbacks:

a) it involves the solution of an m x m non-sparse system of equations so it is quite unattractive when dealing with many control points.

b) to evaluate $Z(x,y)$ at a given point, m logarithms and square roots have to be performed. That can be cumbersome for a large m.

c) this solution considers energy in an infinite plate, and the behaviour at the edges tends to be too "smooth"

The PDE solution for m ground control points deals with an m x m grid, so it is also quite laborious.

V. METHOD PROPOSED

Once the ground control points are found using the SSDA we proceed to obtain a grid on image A that represents the finite element partition used. We want a grid such that no more than 3 points lie in a rectangle. This, although restricting, is a practical way to ensure solvability of the system of equations.

Let us suppose that A is square. We initially partition image A in p x p squares, where p is the smallest integer greater than $\sqrt{m/3}$. The a test is made to see which squares contain more than 3 points. The smallest rectangle containing the three closest points in each square is drawn, and the best partition is chosen among the 4 possible ones generated by the sides of the rectangle (fig.2). The procedure is continued until no more than 3 points lie in a rectangle.

We will idealize image A as a plate structure consisting of $M_1 \times M_2$ elements generated by the grid connected only at nodal points, having $n_1 \times n_2$ nodes with $n_1=M1+1, n_2=M2+1$. We will try to find $M_1 \times M_2$ polynomial functions, one for each element, that will transform image A to B. Each element will be treated independently, but finite element techniques will assure that equilibrium of forces and compatibility of displacements be satisfied at each nodal point.

Each element in A will have three degrees of freedom at each node, namely two rotations and the transverse deflection. Let us denote the lateral deflection by w, the rotation about the x-axis by θx and the rotation about the y-axis by θy. The element then has a total of 12 degrees of freedom.

Displacement in each node i can be expressed as:

$$\{\partial i\} = \begin{Bmatrix} \{\theta xi\} \\ \{\theta yi\} \\ \{wi\} \end{Bmatrix}$$

and the corresponding moments and forces at node i may be written as:

$$\{Fi\} = \begin{Bmatrix} \{Txi\} \\ \{Tyi\} \\ \{Fzi\} \end{Bmatrix}$$

so complete displacement and forces of any element can be written as:

$$\{\partial e\} = \begin{Bmatrix} \{\partial 1\} \\ \{\partial 2\} \\ \{\partial 3\} \\ \{\partial 4\} \end{Bmatrix}$$

and

$$\{Fe\} = \begin{Bmatrix} \{F1\} \\ \{F2\} \\ \{F3\} \\ \{F4\} \end{Bmatrix}$$

As each vector contains 12 elements, stiffness matrix $[Ke]$ such that

$$\{Fe\} = [Ke]\{\partial e\}$$

is a 12 x 12 matrix.

We now have to choose a displacement function $[f(x,y)]$ that defines displacement $\{\partial(x,y)\}$ at each point in the element. θx, θy and w are related by:

$$\theta x = -\partial w/\partial y, \qquad \theta y = \partial w/\partial x$$

so once w is obtained, θx and θy are automatically defined.

As each element has 12 degrees of freedom, we will choose w as a polynomial in twelve unknowns:

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \ldots + \alpha_{12} xy^3 \qquad (1)$$

$$\theta x = -(\alpha_3 + \alpha_5 x + \ldots + 3\alpha_{12} xy^2) \qquad (2)$$

$$\theta y = \alpha_2 + 2\alpha_4 x + \ldots + \alpha_{12} y^3 \qquad (3)$$

It can easily be seen that continuity of w and θx are ensured along the edges where x is constant but θy may be discontinuos along this edge. In general, along any edge a discontinuity of the normal slope can exist, and because of this the function is not ideal. (This can be avoided by introducing into $\{\partial_i\}$ a fourth element Ti called the twist, where $Ti = \partial^2 w/\partial x\partial y$).

Writing equations (1),(2),(3) in matrix form we have

$$\{\partial(x,y)\} = [f(x,y)]\{\alpha\} \qquad (4)$$

where $[f(x,y)]$ is a 3 x 12 matrix and $\{\alpha\}$ is a vector of the 12 unknowns.

Substitution of nodal coordinates in equation (4) leads to

$$\{\partial e\} = [A]\{\alpha\}$$

and solving for $\{\alpha\}$ leads to

$$\{\alpha\} = [A]^{-1}\{\partial e\} \qquad (5)$$

which substituted in (4) leads to

$$\{\partial(x,y)\} = [f(x,y)] \, [A]^{-1} \, \{\partial e\}$$

We have now related displacements within the element in terms of nodal displacements.

Once the needed relations are clearly stated, we are now able to generate the stiffness matrix $[Ke]$ for each element, and the stiffness matrix $[K]$ for the whole system[2]. We also generate the restrictions matrix $[Be]$ for the ground control points substituting its coordinates in (4) and finally obtain the restrictions matrix $[B]$ for the whole system.

$[K]$ is a $(3\ n_1\ n_2) \times (3\ n_1\ n_2)$ banded matrix and $[B]$ is a $(3\ n_1\ n_2\ m)$ sparse matrix.

We want to minimize

$$\{\partial\}' \, [K] \, \{\partial\}$$

where $\{\partial\}$ is the whole systems displacement at each node, subject to

$$[B]\{\partial\} = \{b\}$$

where $\{b\}$ is the vector containg the U_i's.

In other words, we want to minimize

$$\{\partial\}' \, [K] \, \{\partial\} + e(\{\partial\}'[B]' - \{b\}')([B]\{\partial\} - \{b\})$$

where e is a penalty term.

This leads to solving

$$([K] + e[B]'[B])\{\partial\} = e[B]'\{b\}$$

for $\{\partial\}$.

The system is solved using Cholesky triangularization and we obtain $\{\partial\}$. This is done very easily because matrices $[K]$ and $[B]$ are very sparse. Addition of

term $[B]'[B]$ does not diminish sparsity.

Substituting each elements displacements $\{\partial e\}$ in (5), we obtain the $M_1 \times M_2$ polynomial functions necessary to transform image A.

VI. EXAMPLE

We will show below an example used with the present procedure. Although it does not correspond to an image distortion, it is an example already solved by other methods.

The case consists of:

$m=10$ (10 restrictions)
$M_1=M_2=4$ (16 elements)
$n_1=n_2=5$ (25 nodes)

Input data is:

x	y	element	displacement
0.64	0.04	1	2144
0.2	0.44	5	1184
0.52	0.32	7	25712
0.68	0.84	8	2699
0.28	0.2	8	783
0.96	0.2	8	44621
0.48	0.84	9	54610
0.32	0.68	10	869
0.76	0.04	10	4233
0.52	0.52	12	13005

Grid and restrictions are shown in fig. 3.

Matrix formation takes 2 CPU seconds and solution takes 6 CPU seconds. Solution is plotted in fig. 4.

We consider the present method to be very successful for $m \geq 30$.

VII. PROPOSED MODIFICATIONS

Succesive over-relaxation cannot be applied to our method, for the penalty terms cause a bad ratio among the eigenvalues of the matrix $[K] + e[B]'[B]$.

Even though a factorization solution of the system of equations is satisfactory, it is desirable to have an iterative method which will allow us to start with an approximate solution and improve it until it fulfills a convergence criteria.

Two of such approaches have been explored:

a) "Control Problem Approach"

It consists in breaking vector $\{\partial\}$ in a series of n_1 subvectors $\{u(1)\},...,\{u(n_1)\}$, each of them of dimension $3n_2$, and corresponding to the displacements and rotations of the nodes of a row.

The optimization problem, after some algebraic manipulations, can be expressed as to:

minimize
$$1/2 (\sum_{l=1}^{n-1} \{p(i)\}'\{p(i)\}+\{u(1)\}'[W(1)]\{u(1)\}+$$

$$\{u(n_1)\}'[W(n_1)]\{u(n_1)\})$$

subject to

$$\{u(i+1)\}=[A(i)]\{u(i)\}+[B(i)]\{p(i)\}+\{c(i)\}, \quad i=1,...,n1-1$$

$[A]$, $[B]$ and $\{c\}$ are such that for any choice of $\{u(1)\}$ and $\{p(1)\},..., \{p(n1-1)\}$, vectors $\{u(1)\},...,\{u(n1)\}$ fulfill $[A]\{u\}=\{b\}$.

The previous system of equations allows a simple way of computing the gradient with respect to $\{u(i)\}$ which can be used to improve the solution.

b) "Triangular Element Approach"

It consists in using a triangular irregular grid with the vertices corresponding to either control points or to suitable points on the edges of the image.

Even though the approach implies to have

i) a triangle formation routine,
ii) a node ordering procedure,
iii) either some waste in computer memory or a more elaborate way of managing the sparsity of matrices,

the difficulties introduced are less than the extra computer time needed in the control problem approach.

Therefore, work is planned to continue along these lines, adding the capability of working with "twists" (second derivatives specified at nodes).

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IX. LIST OF SYMBOLS

[ ] denotes a matrix.
{ } denotes a vector.
[ ]' denotes a transpose matrix.
{ }' denotes a transpose vector.
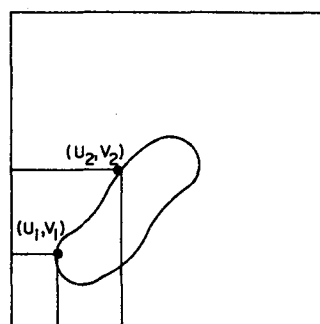lower case letters denote scalars.
$[\ ]^{-1}$ denotes inverse matrix
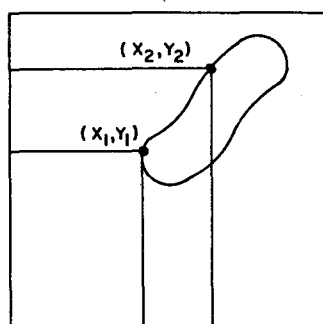
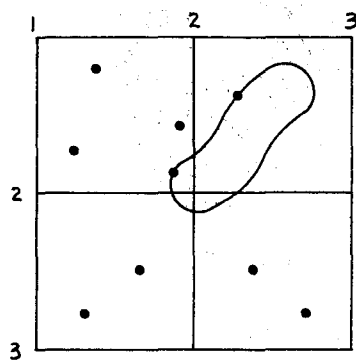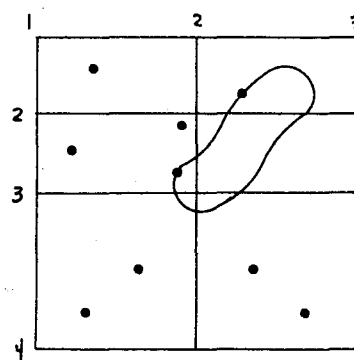FIGURE 1. GROUND CONTROL POINTS FOR IMAGES A AND B.

IMAGE A

FIGURE 2. INITIAL AND FINAL PARTITION DURING GENERATION OF THE GRID.

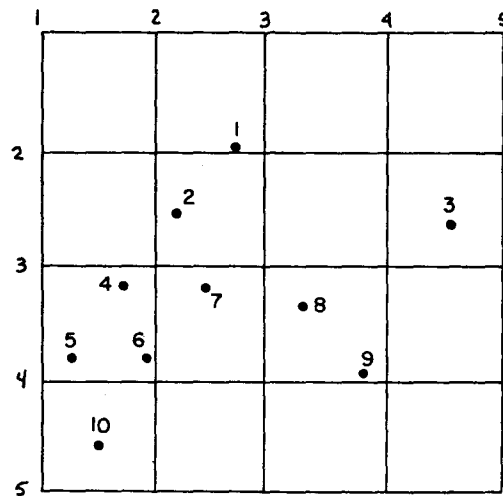CASE $m = 9$, $n_1 = 4$, $n_2 = 3$, $M_1 = 3$, $M_2 = 2$.

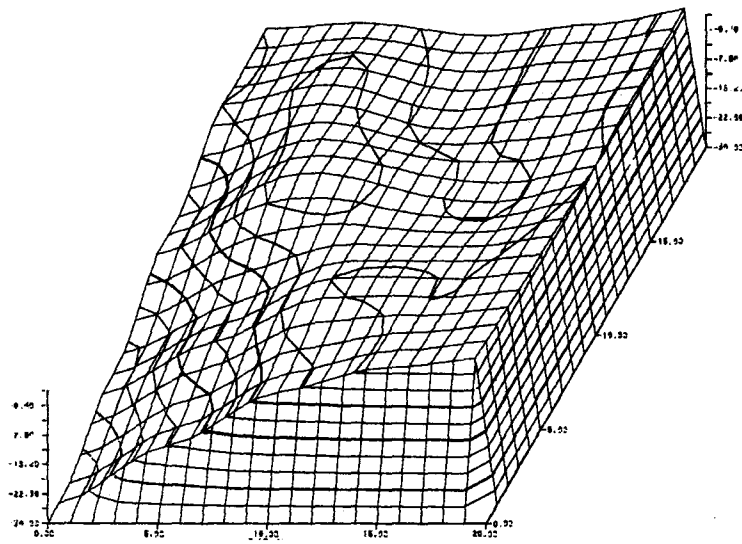FIGURE 3. RESTRICTIONS AND GRID.

FIGURE 4. SOLUTIONS PLOTTED IN THE
Z − AXIS

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